

Critical percolation on random regular graphs

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Abstract

We describe the component sizes in critical independent p -bond percolation on a random d -regular graph on n vertices, where $d \geq 3$ is fixed and n grows. We prove *mean-field* behavior around the critical probability $p_c = \frac{1}{d-1}$.

In particular, we show that there is a scaling window of width $n^{-1/3}$ around p_c in which the sizes of the largest components are roughly $n^{2/3}$ and we describe their limiting joint distribution. We also show that for the subcritical regime, i.e. $p = (1 - \varepsilon(n))p_c$ where $\varepsilon(n) = o(1)$ but $\varepsilon(n)n^{1/3} \rightarrow \infty$, the sizes of the largest components are concentrated around an explicit function of n and $\varepsilon(n)$ which is of order $o(n^{2/3})$. In the supercritical regime, i.e. $p = (1 + \varepsilon(n))p_c$ where $\varepsilon(n) = o(1)$ but $\varepsilon(n)n^{1/3} \rightarrow \infty$, the size of the largest component is concentrated around the value $\frac{2d}{d-2}\varepsilon(n)n$ and a duality principle holds: other component sizes are distributed as in the subcritical regime.

1 Introduction

Let $d \geq 3$ be a fixed integer, $n > 0$ an integer such that dn is even, and $p \in (0, 1)$. Let $G(n, d, p)$ be a random graph on n vertices obtained by drawing uniformly a random d -regular graph on n vertices and then performing independent p -bond percolation on it, i.e., we independently retain each edge with probability p and delete it with probability $1 - p$. Alon, Benjamini and Stacey proved in [2] that the model $G(n, d, \frac{c}{d-1})$ exhibits a phase transition as c grows: the cardinality of the largest component \mathcal{C}_1 is of order $\log n$ for $c < 1$ and of order n for $c > 1$.

Recall that similar behavior is exhibited in the random graph $G(n, p)$, introduced by Erdős and Rényi [13]. They discovered that as c grows,

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$G(n, c/n)$ exhibits a *double jump*: the cardinality of the largest component \mathcal{C}_1 is of order $\log n$ for $c < 1$, of order $n^{2/3}$ for $c = 1$ and linear in n for $c > 1$. In fact, for the critical case $c = 1$ the argument in [13] only established the lower bound; the upper bound was proved much later in [7], [17] and [18]; see also [21] for a simple proof of this upper bound. These works established the existence of a “scaling-window” of width $n^{-1/3}$ around the point $\frac{1}{n}$, i.e., for all p of the form $\frac{1}{n}(1 + O(n^{-1/3}))$ the random variable $|\mathcal{C}_1|/n^{2/3}$ converges in distribution to a non-trivial random variable, and in particular, is not concentrated. Furthermore, outside of this scaling window, i.e. for p of the form $\frac{1}{n}(1 + \varepsilon(n))$ where $\varepsilon(n) = o(1)$ but $\varepsilon(n)n^{1/3} \rightarrow \infty$, the random variable $|\mathcal{C}_1|$ is concentrated around some known value. This is often called “mean-field” behavior around the critical probability $p_c(n) = \frac{1}{n}$.

Itai Benjamini (personal communication) asked whether percolation on a random d -regular graph has mean-field behavior. In this paper we answer his question affirmatively for d fixed and n growing, and give a complete description of the component sizes at criticality. We establish the existence of a scaling window of width $n^{-1/3}$ around the critical probability $\frac{1}{d-1}$ (in which component sizes have a non-trivial limiting distribution) and show that outside the window the largest component (and the ℓ -th largest component as well) is concentrated. Boris Pittel (personal communication) informed us that he had obtained similar (but somewhat less precise) results.

Recall (see [8] and [15]) that in the Erdős-Rényi random graph $G(n, \frac{1-\varepsilon(n)}{n})$, where $\varepsilon(n) > 0$ satisfies $\varepsilon(n) \rightarrow 0$ and $\varepsilon(n)n^{1/3} \rightarrow \infty$, for any fixed integer $\ell > 0$ we have

$$\frac{|\mathcal{C}_\ell|}{\psi_n(\varepsilon(n))} \xrightarrow{P} 1 \quad \text{as } n \rightarrow \infty, \quad (1)$$

where

$$\psi_n(\varepsilon) = 2\varepsilon^{-2} \log(n\varepsilon^{-3}). \quad (2)$$

The following proposition provides general upper bounds on the size of the largest component which are valid for *all* d -regular graphs. In particular, part 1 provides an upper bound on $|\mathcal{C}_1|$ in the subcritical regime, similar to the one implied in (1), and part 2 and 3 provide upper bounds for other regimes of p .

Proposition 1 [General upper bounds] *Let G be a d -regular graph for $d \geq 3$ and denote by $\mathcal{C}_1(G_p)$ the largest connected component of the random graph obtained by bond percolation on G with probability p . We have*

1. If $p = \frac{1-\varepsilon(n)}{d-1}$ where $\varepsilon(n) \geq 0$ is a sequence such that $\varepsilon(n) \rightarrow 0$ and $\varepsilon(n)n^{1/3} \rightarrow \infty$, then for any $\eta > 0$

$$\mathbf{P}\left(|\mathcal{C}_1(G_p)| > (1+\eta)\frac{d-2}{d-1}\psi_n(\varepsilon(n))\right) \rightarrow 0,$$

as $n \rightarrow \infty$.

2. If $p \leq \frac{1}{d-1}$ where $\lambda \leq 0$ then for any $A > 1$

$$\mathbf{P}\left(|\mathcal{C}_1(G_p)| > An^{2/3}\right) \leq \frac{8}{A^{3/2}}.$$

3. There exists a constant $C > 0$ such that if $p = \frac{1+\varepsilon(n)}{d-1}$ where $\varepsilon(n) > 0$ then

$$\mathbf{E} |\mathcal{C}_1(G_p)| \leq C(n^{2/3} + \varepsilon(n)n).$$

For a random regular graph, we can sharpen these upper bounds and prove corresponding lower bounds. In the following we denote by $\{\mathcal{C}_j\}_{j \geq 1}$ the connected components of $G(n, d, p)$ ordered in decreasing size. We emphasize that all the theorems apply for d fixed and n growing. See Section 9 for further discussion on the case where d grows with n .

Theorem 2 [Critical window bounds] Consider $G(n, d, p)$ with $p = \frac{1+\lambda n^{-1/3}}{d-1}$ for some $\lambda \in \mathbb{R}$ where $d \geq 3$ is fixed. Then there exist constants $c(\lambda, d) > 0$ and $C(\lambda, d) < \infty$ such that for any $A > 0$ and all n ,

$$\mathbf{P}(|\mathcal{C}_1| \geq An^{2/3}) \leq \frac{C(\lambda, d)e^{-c(\lambda, d)A^3}}{A}. \quad (3)$$

Furthermore, there exists a constant $D = D(\lambda, d)$ such that for $\delta > 0$ small enough and all n ,

$$\mathbf{P}\left(|\mathcal{C}_1| < \lceil \delta n^{2/3} \rceil\right) \leq D(\lambda, d)\delta^{1/2}. \quad (4)$$

The next two theorems describe the largest component behavior outside of the scaling window. In particular, outside the scaling window, the largest component is concentrated; however, the structure of the graphs is quite different depending on whether we are above or below the scaling window. Above the window the largest component is of order $\varepsilon(n)n$ and it is the unique component of this size. Below the window, the largest component is of order $\varepsilon^{-2}(n)\log(n\varepsilon^3)$, but so is the ℓ -th largest component, for any fixed $\ell > 1$. The following theorem provides the analogous statement to (1) for $G(n, d, p)$.

Theorem 3 [Below the critical window] Recall the definition of ψ_n from (2) and let $\varepsilon(n) > 0$ be a sequence such that $\varepsilon(n) \rightarrow 0$ and $\varepsilon(n)n^{1/3} \rightarrow \infty$. Consider $G(n, d, p)$ with $p = \frac{1-\varepsilon(n)}{d-1}$ where $d \geq 3$ is fixed, then for any fixed integer $\ell > 0$ we have

$$\frac{|\mathcal{C}_\ell|}{\psi_n(\varepsilon(n))} \xrightarrow{P} \frac{d-2}{d-1} \quad \text{as } n \rightarrow \infty, \quad (5)$$

We now turn to the supercritical case. In the Erdős-Rényi random graph $G(n, \frac{1+\varepsilon(n)}{n})$, where $\varepsilon(n) > 0$ satisfies $\varepsilon(n) \rightarrow 0$ and $\varepsilon(n)n^{1/3} \rightarrow \infty$ we have (see [7], [17] and also [22])

$$\frac{|\mathcal{C}_1|}{2n\varepsilon(n)} \xrightarrow{P} 1 \quad \text{as } n \rightarrow \infty,$$

and (1) holds for any fixed integer $\ell > 1$, controlling the size of the smaller components. The following theorem provides the analogous statement for $G(n, d, p)$.

Theorem 4 [Above the critical window] Let $\varepsilon(n) > 0$ be a sequence such that $\varepsilon(n) \rightarrow 0$ and $\varepsilon(n)n^{1/3} \rightarrow \infty$. Consider $G(n, d, p)$ with $p = \frac{1+\varepsilon(n)}{d-1}$, where $d \geq 3$ is fixed, then

$$\frac{|\mathcal{C}_1|}{2n\varepsilon(n)} \xrightarrow{P} \frac{d}{d-2} \quad \text{as } n \rightarrow \infty.$$

Furthermore, for any fixed integer $\ell > 1$ we have that (5) holds, controlling the size of \mathcal{C}_ℓ .

Next we turn to describe the limiting distribution of the component sizes inside the scaling window $p = \frac{1+\lambda n^{-1/3}}{d-1}$, in an analogous way to [1]. Let $\{B(s) : s \in [0, \infty)\}$ be standard Brownian motion and for $\lambda \in \mathbb{R}$ define the process

$$B^\lambda(s) = B\left(\frac{(d-2)}{(d-1)}s\right) + \lambda s - \frac{(d-2)}{2d}s^2, \quad s \in [0, \infty). \quad (6)$$

Also, consider the reflected process

$$W^\lambda(s) = B^\lambda(s) - \min_{0 \leq s' \leq s} B^\lambda(s'). \quad (7)$$

An excursion γ of W^λ is a time interval $[l(\gamma), r(\gamma)]$ in which $W^\lambda(l(\gamma)) = W^\lambda(r(\gamma)) = 0$, and $W^\lambda(s) > 0$ for all $l(\gamma) < s < r(\gamma)$. The excursion has length $|\gamma| = r(\gamma) - l(\gamma)$. The sequence $(|\gamma_j|)_{j \geq 1}$ of excursion lengths, in decreasing order, is a random variable in ℓ^2 almost surely (see [1]).

Theorem 5 [Limiting distribution] Fix $\lambda \in \mathbb{R}$ and let $p = \frac{1+\lambda n^{-1/3}}{d-1}$, where $d \geq 3$ is fixed, then

$$n^{-2/3} \cdot (|\mathcal{C}_1|, |\mathcal{C}_2|, \dots) \xrightarrow{d} (|\gamma_j|)_{j \geq 1},$$

where convergence holds with respect to the ℓ^2 norm.

In [23], the authors prove that in bond percolation on any d -regular graph on n vertices with $p \leq \frac{1+\lambda n^{-1/3}}{d-1}$, if the resulting graph typically has components of size $n^{2/3}$ then their diameter is of order $n^{1/3}$ and the mixing time of the lazy simple random walk on these components is of order n . See [23] for more details and definitions. The following is an immediate corollary of Theorem 5 above and Theorem 1.2 of [23].

Corollary 6 Consider $G(n, d, p)$ with $p = \frac{1+\lambda n^{-1/3}}{d-1}$ for some $\lambda \in \mathbb{R}$, where $d \geq 3$ is fixed. Denote by $\text{diam}(\mathcal{C}_\ell)$ the diameter of \mathcal{C}_ℓ and let $T_{\text{mix}}(\mathcal{C}_\ell)$ be the mixing time of the lazy simple random walk on \mathcal{C}_ℓ . Then for any fixed integer $\ell > 0$ and any $\varepsilon > 0$ there exists $A = A(\varepsilon, \lambda, \ell) < \infty$ such that for all large n ,

- $\mathbf{P}\left(\text{diam}(\mathcal{C}_\ell) \notin [A^{-1}n^{1/3}, An^{1/3}]\right) < \varepsilon,$
- $\mathbf{P}\left(T_{\text{mix}}(\mathcal{C}_\ell) \notin [A^{-1}n, An]\right) < \varepsilon.$

A major challenge is to give criteria for specific d -regular graphs to exhibit mean-field behavior (the theorems of this paper establish that this occurs for most d -regular graphs). Substantial progress in this direction was made in [9] and [10].

The rest of the paper is organized as follows. As the proof of Proposition 1 is simple and instructive, we provide it in Section 2. In Section 3 we describe a discrete exploration process which generates a random sample of $G(n, d, p)$. The analysis of this process is crucial for proving the results of this paper and is presented in Section 4. From there we proceed to prove Theorem 2 in Section 5. Theorems 3 and 4, describing the behavior above and below the scaling window, are proved in Section 6 and 7 respectively. Theorem 5 is proved in Section 8 and we end with some concluding remarks in Section 9.

We use the standard asymptotic notation. For two functions $f(n)$ and $g(n)$, we write $f = o(g)$ if $\lim_{n \rightarrow \infty} f/g = 0$. Also, $f = O(g)$ if there exists

an absolute constant $C > 0$ such that $f(n) < Cg(n)$ for all large enough n and $f = \Theta(g)$ if both $f = O(g)$ and $g = O(f)$ hold.

2 Proof of the general upper bounds (Prop. 1)

For the proofs in this section and in sections to follow we present some standard facts about processes with independent increments.

Lemma 7 *Let β be a random variable supported on the integers with $\mathbf{P}(\beta < -1) = 0$. Let $\{\beta_i\}$ be i.i.d. random variables distributed as β and let $W_t = W_0 + \sum_{i=1}^t \beta_i$, where $W_0 > 0$ is some integer. For an integer $h > W_0$ define the stopping time*

$$\gamma_h = \min_t \{W_t = 0 \text{ or } W_t \geq h\}.$$

We have

(i) *If $c > 0$ is such that $\mathbf{E} e^{-c\beta} \geq 1$ then*

$$\mathbf{P}(W_{\gamma_h} > 0) \leq \frac{1 - e^{-cW_0}}{1 - e^{-ch}}.$$

(ii) *If $c > 0$ is such that $\mathbf{E} e^{c\beta} \leq 1$ then*

$$\mathbf{P}(W_{\gamma_h} > 0) \leq \frac{e^{cW_0} - 1}{e^{ch} - 1}.$$

Proof. This is a standard application of the optional stopping theorem (see [12]). The assumption $\mathbf{E} e^{-c\beta} \geq 1$ implies that $\{e^{-cW_t}\}$ is a submartingale. Optional stopping gives

$$e^{-cW_0} \leq 1 - \mathbf{P}(W_{\gamma_h} > 0) + e^{-ch} \mathbf{P}(W_{\gamma_h} > 0),$$

which yields assertion (i) of the lemma. The assumption $\mathbf{E} e^{c\beta} \leq 1$ implies that $\{e^{cW_t}\}$ is a supermartingale, and similarly we get assertion (ii) of the lemma. \square

The following lemma is a variant of a lemma due to Bahadur and Rao [11].

Lemma 8 *Let β be a non-lattice, integer valued random variable with $\mathbf{E} \beta^2 < \infty$. Let $\{\beta_i\}$ be i.i.d. random variables distributed as β and let $W_t = W_0 + \sum_{i=1}^t \beta_i$. Let τ be the hitting time of 0, i.e.*

$$\tau = \min_t \{W_t = 0\}.$$

If $\theta_0 > 0$ satisfies

$$\mathbf{E}[\beta e^{\theta_0 \beta}] = 0, \quad (8)$$

then for any integer $\ell > 0$ we have

$$\mathbf{P}(\tau = \ell) = \Theta\left(\ell^{-3/2} \varphi(\theta_0)^\ell\right),$$

where $\varphi(\theta) = \mathbf{E}e^{\theta\beta}$, and the constants in the Θ depend only on β and W_0 but not on ℓ .

For the proof of Lemma 8 we require the following variant of a lemma due to Spitzer [25]. For completeness, we include its proof here.

Lemma 9 *Let $a_0, \dots, a_{k-1} \in \mathbb{Z}$ be such that $\sum_{i=0}^{k-1} a_i = -d$. Then there are at least one and at most d numbers $j \in \{0, \dots, k-1\}$ such that for all $\ell \in \{0, \dots, k-2\}$*

$$\sum_{i=0}^{\ell} a_{(j+i) \bmod k} > -d.$$

Proof. Continue the sequence periodically such that $a_{k+s} = a_s$ for any integer $s > 0$. Let j be the first global minimum of the function $f(j) = \sum_{i=0}^j a_i$ on the domain $\{0, \dots, k-1\}$. It is easy to see that for that j , and any $\ell \in \{0, \dots, k-2\}$

$$\sum_{i=0}^{\ell} a_{(j+i)} > -d.$$

Assume now that there were $j_1 < \dots < j_{d+1}$ all in $\{0, \dots, k-1\}$ satisfying that for all $\ell \in \{0, \dots, k-2\}$

$$\sum_{i=0}^{\ell} a_{(j_r+i)} > -d,$$

for all $r \in \{1, \dots, d+1\}$. Define a function $g(r)$ on $\{1, \dots, d+1\}$ by

$$g(1) = \sum_{i=j_{d+1}}^{k-1+j_1-1} a_i,$$

$$g(r) = \sum_{i=j_{r-1}}^{j_r-1} a_i, \quad r \in \{2, \dots, d+1\}.$$

As $\sum_{i=j_{r-1}}^{k-1+j_{r-1}} a_i = -d$ and $\sum_{i=j_r}^{k-1+j_r-1} a_i > -d$ we find that $g(r) \leq -1$ for all $r \in \{1, \dots, d+1\}$. The assumption $\sum_{i=0}^{k-1} a_i = -d$ implies that $g(1) + \dots + g(d+1) = -d$ and we have arrived at a contradiction. \square

Proof of Lemma 8. Let β_θ be a random variable distributed as

$$\mathbf{P}(\beta_\theta = t) = \varphi(\theta)^{-1} e^{\theta t} \mathbf{P}(\beta = t).$$

Let $\{\beta_\theta(i)\}$ be a sequence of i.i.d. random variables distributed as β_θ and let $W_\theta(t) = W_0 + \sum_{i=1}^t \beta_\theta(i)$. Let $I = \{(t_1, \dots, t_\ell) : W_0 + t_1 + \dots + t_\ell = 0\}$ and observe that

$$\frac{\mathbf{P}(W_\ell = 0)}{\mathbf{P}(W_\theta(\ell) = 0)} = \frac{\sum_{(t_1, \dots, t_\ell) \in I} \prod_{i=1}^\ell \mathbf{P}(\beta_i = t_i)}{\varphi(\theta)^{-\ell} \sum_{(t_1, \dots, t_\ell) \in I} e^{\theta(t_1 + \dots + t_\ell)} \prod_{i=1}^\ell \mathbf{P}(\beta_i = t_i)} = e^{\theta W_0} \varphi(\theta)^\ell.$$

We now take $\theta = \theta_0$. By (8) we have that $\mathbf{E} \beta_{\theta_0} = 0$, thus by the local central limit theorem (see [12], Section 2.5) we have that $\mathbf{P}(W_{\theta_0}(\ell) = 0) = \Theta(\ell^{-1/2})$. Thus,

$$\mathbf{P}(W_\ell = 0) = \Theta(\ell^{-1/2} \varphi(\theta_0)^\ell),$$

and by Lemma 9 we learn that $\mathbf{P}(\tau = \ell) = \Theta(\ell^{-1}) \mathbf{P}(W_\ell = 0)$, concluding our proof. \square

Proof of Proposition 1. For a graph G , denote by G_p the random graph obtained by bond percolation on G with probability p . For a vertex v and let $\mathcal{C}(v)$ denote the connected component that contains v in G_p . We recall an exploration process, developed independently by Martin-Löf [19] and Karp [16]. In this process, vertices will be either *active*, *explored* or *neutral*. At each time t , the number of active vertices will be denoted Y_t and the number of explored vertices will be t . Fix an ordering of the vertices, with v first. As an upper bound, assume some edge (v, u) adjacent to v is open. At time $t = 0$, the vertices v and u are active and all other vertices are neutral, so $Y_0 = 2$. In step $t > 0$ let w_t be the first active vertex. Denote by η_t the number of neutral neighbors of w_t in G_p and change the status of these vertices to *active*. Then, set w_t itself *explored*. The process stops when Y_t hits 0, and observe that since at each step we set precisely one vertex explored we have $|\mathcal{C}(v)| \leq \min\{t : Y_t = 0\}$. Let $\{w_1, w_2, \dots\}$ be independent random variables distributed as $\text{Bin}(d-1, p) - 1$. Let $W_t = 2 + \sum_{i=1}^t w_i$. As G is d -regular, it is clear that we can couple the process $\{Y_t\}$ and $\{W_t\}$ such that $Y_t \leq W_t$ for all $t \leq |\mathcal{C}(v)|$.

We begin with the proof of part 1 of the Theorem. We will use Lemma 8 with $\beta = w - 1$, where w is distributed as $\text{Bin}(d - 1, p)$ and $p = \frac{1-\varepsilon}{d-1}$. If we write $w = \sum_{j=1}^{d-1} I_j$ where I_j are i.i.d. Bernoulli(p) random variables, we get that for any $\theta > 0$

$$\mathbf{E} w e^{\theta w} = (d-1) \mathbf{E} \left[\prod_{j=2}^{d-1} e^{\theta I_j} \right] \mathbf{E} I_1 e^{\theta I_1} = (d-1) p e^{\theta} (1-p + e^{\theta} p)^{d-2}.$$

As $\mathbf{E} e^{\theta w} = (1-p + p e^{\theta})^{d-1}$ we have

$$\mathbf{E} \beta e^{\theta \beta} = e^{-\theta} (1-p + e^{\theta} p)^{d-2} [p(d-1)e^{\theta} - (1-p + p e^{\theta})].$$

Let $\theta_0 > 0$ be a number such that $\mathbf{E} \beta e^{\theta_0 \beta} = 0$, then by estimating $e^x = 1 + x + O(x^2)$ in the last equation we find that

$$p(d-2)(1+\theta_0) + p + O(\theta_0^2) = 1,$$

thus

$$\theta_0 = \frac{(d-1)\varepsilon}{d-2} + O(\varepsilon^2).$$

For any $\theta > 0$ by estimating $e^x = 1 + x + x^2/2 + O(x^3)$ we get

$$\begin{aligned} \varphi(\theta) &= \mathbf{E} e^{\theta \beta} = e^{-\theta} (1 + p(e^{\theta} - 1))^{d-1} \\ &= \left(1 - \theta + \theta^2/2\right) \left(1 + (d-1)p(\theta + \theta^2/2) + \frac{(d-1)(d-2)p^2\theta^2}{2}\right) + O(\theta^3). \end{aligned}$$

By simplifying and plugging in the value of θ_0 we find that

$$\varphi(\theta_0) = 1 - \frac{(d-1)\varepsilon^2}{2(d-2)} + O(\varepsilon^3).$$

Let $\tau = \min\{t : W_t = 0\}$ then Lemma 8 implies that

$$\mathbf{P}(\tau > T) \leq \sum_{\ell=T+1}^{\infty} O\left(\ell^{-3/2} \left(1 - \frac{(d-1)\varepsilon^2}{2(d-2)} + O(\varepsilon^3)\right)^{\ell}\right).$$

We take

$$T = (1+\eta) \frac{2(d-2)}{d-1} \varepsilon^{-2} \log(n\varepsilon^3),$$

and a straightforward computation using $1-x \leq e^{-x}$ yields that for some fixed $c > 0$

$$\mathbf{P}(\tau > T) \leq O\left(\varepsilon(n\varepsilon^3)^{-(1+\eta)(1-c\varepsilon)} \log(n\varepsilon^3)^{-3/2}\right).$$

As $Y_t \leq W_t$ for all $t \leq |\mathcal{C}(v)|$ we have $\mathbf{P}(|\mathcal{C}(v)| > T) \leq \mathbf{P}(\tau > T)$. Denote by X the number of vertices v of G such that $|\mathcal{C}(v)| > T$. If $|\mathcal{C}_1| > T$ then $X > T$. We conclude that for some $c_1 > 0$

$$\begin{aligned} \mathbf{P}(|\mathcal{C}_1| > T) &\leq \mathbf{P}(X > T) \leq \frac{\mathbf{E} X}{T} \leq \frac{n\mathbf{P}(|\mathcal{C}(v_1)| > T)}{T} \\ &\leq \frac{Cn\varepsilon(n\varepsilon^3)^{-(1+\eta)(1-c\varepsilon)}}{\varepsilon^{-2} \log^{5/2}(n\varepsilon^3)} \leq (n\varepsilon^3)^{-\eta(1-c_1\varepsilon)+c_1\varepsilon} \rightarrow 0, \end{aligned}$$

which concludes part 1 of the proposition.

We now prove part 2 of the Proposition, following the strategy laid out in [21]. By monotonicity we may assume that $p = \frac{1}{d-1}$. In that case $\{W_t\}$ is a martingale with $\mathbf{E} W_0 \leq 2$. Define γ_h as in Lemma 7, so by optional stopping we get that $\mathbf{E} W_0 = \mathbf{E} W_{\gamma_h} \geq h\mathbf{P}(W_{\gamma_h} > 0)$, whence

$$\mathbf{P}(W_{\gamma_h} > 0) \leq \frac{2}{h}. \quad (9)$$

By Corollary 6 in [21] (see also inequality (3) of [21]) we also have

$$\mathbf{E}[W_{\gamma_h}^2 \mid W_{\gamma_h} > 0] \leq h^2 + 3h. \quad (10)$$

It is immediate to verify that $W_t^2 - (1 - \frac{1}{d-1})t$ is also a martingale. Optional stopping, (9) and (10) gives that

$$(1 - \frac{1}{d-1})\mathbf{E} \gamma_h \leq \mathbf{E} W_{\gamma_h}^2 = \mathbf{P}(W_{\gamma_h} > 0)\mathbf{E}[W_{\gamma_h}^2 \mid W_{\gamma_h} > 0] \leq 2h + 6.$$

As $d > 2$ we get

$$\mathbf{E} \gamma_h \leq 4h + 12. \quad (11)$$

Hence as long as $h > 12$

$$\mathbf{P}(\gamma_h \geq h^2) \leq \frac{5}{h}.$$

Denote $\gamma_h^* = \gamma_h \wedge h^2$. By the previous inequality and (9), we have

$$\mathbf{P}(W_{\gamma_h^*} > 0) \leq \mathbf{P}(W_{\gamma_h} > 0) + \mathbf{P}(\gamma_h \geq h^2) \leq \frac{7}{h}.$$

Let $T = h^2$ and observe that if $|\mathcal{C}(v)| > h^2$ we must have $W_{\gamma_h^*} > 0$, thus $\mathbf{P}(|\mathcal{C}(v)| > T) \leq \frac{7}{\sqrt{T}}$. Again denote by X the number of vertices v of G such

that $|\mathcal{C}(v)| > T$. If $|\mathcal{C}_1| > T$ then $X > T$. We put $T = \left(\lfloor \sqrt{An^{2/3}} \rfloor\right)^2$ and conclude that

$$\mathbf{P}(|\mathcal{C}_1| > T) \leq \frac{\mathbf{E} X}{T} \leq \frac{7n}{T^{3/2}} \leq \frac{8}{A^{3/2}},$$

for large enough n , as required.

We now prove part 3 of the Theorem. For an integer $k > 0$ denote by X_k the number of vertices v of G such that $|\mathcal{C}(v)| > k$. It is clear that if $|\mathcal{C}_1| > k$ then $|\mathcal{C}_1| \leq X_k$, thus

$$\mathbf{E} |\mathcal{C}_1(G_p)| \leq k + \mathbf{E} X_k. \quad (12)$$

We estimate the last term of the previous display in a similar way to the proof of part 1 of the proposition. Put $p = \frac{1+\varepsilon}{d-1}$, let w be distributed as $\text{Bin}(d-1, p)$ and $\beta = w - 1$. By an almost identical calculation to the one done in part 1 we get that in the notation of Lemma 8

$$\theta_0 = -\frac{(d-1)\varepsilon}{d-2} + O(\varepsilon^2),$$

and

$$\varphi(\theta_0) = 1 - \frac{(d-1)\varepsilon^2}{2(d-2)} + O(\varepsilon^3).$$

Lemma 8 and our usual coupling gives that for some $C > 0$,

$$\mathbf{P}(|\mathcal{C}(v)| > k) \leq \mathbf{P}(\tau > k) \leq \sum_{\ell > k} C\ell^{-3/2}(1 - \Theta(\varepsilon^2))^\ell.$$

A straightforward calculation with the sum in the previous display shows we can bound it from above by $C(k^{-1/2} + \varepsilon)$ for some fixed $C > 0$. We find that $\mathbf{E} X_k = n\mathbf{P}(|\mathcal{C}(v)| > k) \leq C\varepsilon n + Cnk^{-1/2}$. Choosing $k = n^{2/3}$ and plugging into (12) concludes the proof. \square

3 The random regular graph and the exploration process

The following model, known as the *configuration model*, was introduced by Bollobás in [6] (see also [4] and [26]) and was used to construct a uniform random d -regular graph on n vertices, assuming dn is even. Consider the vertex set $\{1, \dots, dn\}$ as n distinct d -tuples. Draw a uniform perfect matching

on the set $\{1, \dots, dn\}$, and then contract every d -tuple into a single vertex. It was shown in [6] and [4] that with probability tending to $\exp(\frac{1-d^2}{4})$ as $n \rightarrow \infty$ this process yields a simple d -regular graph. Moreover, conditioning on this event, the graph obtained is uniformly distributed among all *simple* d -regular graphs on n vertices.

A uniform perfect matching on a set can be obtained by drawing the edges of the matching sequentially: for each edge choose the first vertex according to any rule (deterministic or random) and then choose the second vertex uniformly at random among the unmatched vertices. This motivates exploring the connected components (in the spirit of [16] and [19]) by drawing a uniform matching on $\{1, \dots, dn\}$ sequentially, and independently percolating each edge of the matching; we call this process the *exploration process*. In this process, vertices will be either *active*, *explored* or *neutral* and each d -tuple may contain vertices with different status. Choose an ordering of the vertices $\{v_{i,k} : 1 \leq i \leq n, 1 \leq k \leq d\}$ where $\{v_{i,1}, \dots, v_{i,d}\}$ is the i -th d -tuple, for $0 \leq k \leq n-1$. Initially, the first d -tuple, vertices $\{v_{1,1}, \dots, v_{1,d}\}$, are active and all other vertices are neutral. At each time $t > 0$, if there are active vertices, let w_t be the first active vertex; if there are no active vertices, let w_t be the next neutral vertex and change the status of the neutral vertices in w_t 's d -tuple to *active* (including the status of w_t itself). Now match w_t with a uniformly drawn unmatched vertex η_t . If η_t is neutral and the edge (w_t, η_t) is retained in the percolation then we change the status of the neutral vertices in η_t 's d -tuple to *active*, and we also set w_t and η_t *explored*. If η_t is neutral and the edge (w_t, η_t) is not retained in the percolation or if η_t is active, just set w_t and η_t *explored* without changing the status of any other vertex. This gives a graph on $\{v_1, \dots, v_{dn}\}$; we obtain the multi-graph $G^*(n, d, p)$ on n vertices by contracting each d -tuple to a single vertex. Denote by *Simple* the event that the perfect matching constructed by the exploration process yields a simple d -regular graph. By [6] and our previous discussion we have

$$\mathbf{P}(\text{Simple}) = \exp\left(\frac{1-d^2}{4}\right) + o(1), \quad (13)$$

and by our previous discussion, if we condition on this event, then $G^*(n, d, p)$ is distributed as $G(n, d, p)$.

In order to analyze the exploration process we introduce the following random variables. For $0 \leq k \leq d$ and $t \leq dn/2$ denote by $\mathbf{N}_t^{(k)}$ the set of d -tuples which have precisely k neutral vertices *after* η_t was drawn and *before*

w_{t+1} is chosen, and by $\tilde{\mathbf{N}}_t^{(k)}$ the set of d -tuples which have precisely k neutral vertices *after* w_{t+1} was chosen and *before* η_{t+1} is drawn. Let $N_t^{(k)}$ and $\tilde{N}_t^{(k)}$ denote the cardinality of these sets, respectively. For a vertex v , denote by $[v]$ the tuple containing v . Hence, the notation $[w_{t+1}] \in \mathbf{N}_t^{(k)}$ implies that the d -tuple of w_{t+1} , after w_{t+1} was chosen, has precisely k neutral vertices, and therefore w_{t+1} was chosen neutral (i.e., there were no active vertices remaining). Similarly, $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}$ is the *event* that the d -tuple of η_t has k neutral vertices after η_t was drawn, and that η_t was drawn neutral. For an edge e we write $e \in G_p$ to denote that e was retained in the percolation.

The exploration process dictates the recursive dynamics of these random variables. The number of d -tuples which have d neutral vertices after w_1 is chosen is $n - 1$; at each time $t > 0$ we have $N_t^{(d)} = \tilde{N}_{t-1}^{(d)} - 1$ if $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(d)}$ and $\tilde{N}_t^{(d)} = N_t^{(d)} - 1$ if $[w_{t+1}] \in \mathbf{N}_t^{(d)}$. Hence,

$$\tilde{N}_0^{(d)} = n - 1,$$

$$N_t^{(d)} = \tilde{N}_{t-1}^{(d)} - \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(d)}\}}, \quad \tilde{N}_t^{(d)} = N_t^{(d)} - \mathbf{1}_{\{[w_{t+1}] \in \mathbf{N}_t^{(d)}\}}. \quad (14)$$

For $0 < k < d$, at time $t = 0$ there are no d -tuples with k neutral vertices. At each time $t > 0$ we have $N_t^{(k)} = \tilde{N}_{t-1}^{(k)} - 1$ if $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}$ and $N_t^{(k)} = \tilde{N}_{t-1}^{(k)} + 1$ if $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k+1)}$ and the edge (w_t, η_t) is *not* retained in the percolation. We also have $\tilde{N}_t^{(k)} = N_t^{(k)} - 1$ if $[w_{t+1}] \in \mathbf{N}_t^{(k)}$. Hence,

$$\tilde{N}_0^{(k)} = 0,$$

$$N_t^{(k)} = \tilde{N}_{t-1}^{(k)} - \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}\}} + \mathbf{1}_{\{(w_t, \eta_t) \notin G_p\}} \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k+1)}\}}. \quad (15)$$

$$\tilde{N}_t^{(k)} = N_t^{(k)} - \mathbf{1}_{\{[w_{t+1}] \in \mathbf{N}_t^{(k)}\}}, \quad (16)$$

Finally we have $\tilde{N}_0^{(0)} = 1$ (as the d -tuple of w_1 has no neutral vertices) and at each time $t > 0$ we have $N_t^{(0)} = \tilde{N}_{t-1}^{(0)} + 1$ if η_t is drawn neutral and the edge (w_t, η_t) was retained in the percolation, or if $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(1)}$ and the edge (w_t, η_t) was *not* retained in the percolation. We also have $\tilde{N}_t^{(0)} = N_t^{(0)} + 1$ if w_{t+1} is chosen neutral, i.e., in the case where no more active vertices are left. Hence,

$$\tilde{N}_0^{(0)} = 1,$$

$$\begin{aligned}
N_t^{(0)} = \tilde{N}_{t-1}^{(0)} &+ \mathbf{1}_{\{(w_t, \eta_t) \in G_p\}} \mathbf{1}_{\{\eta_t \text{ drawn neutral}\}} \\
&+ \mathbf{1}_{\{(w_t, \eta_t) \notin G_p\}} \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(1)}\}}, \tag{17}
\end{aligned}$$

$$\tilde{N}_t^{(0)} = N_t^{(0)} + \mathbf{1}_{\{w_{t+1} \text{ chosen neutral}\}}. \tag{18}$$

Denote by \mathbf{A}_t the set of active vertices *after* η_t was drawn and *before* w_{t+1} is chosen and by $\tilde{\mathbf{A}}_t$ the set of active vertices *after* w_{t+1} was chosen and *before* η_{t+1} is drawn. Let A_t and \tilde{A}_t denote the cardinality of these sets, respectively. Let $\{\xi_t\}$ be random variables defined by

$$\xi_t = \mathbf{1}_{\{(w_t, \eta_t) \in G_p\}} \sum_{k=2}^d (k-1) \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}\}} - \mathbf{1}_{\{\eta_t \in \tilde{\mathbf{A}}_{t-1}\}} - 1. \tag{19}$$

For the vertex w_t denote by $N(w_t)$ the number of neutral vertices in $[w_t]$ *after* w_t was chosen and *before* η_t was drawn, including w_t itself. Note that if w_t is active then $N(w_t) = 0$, so this number is non-zero only if w_t is neutral, i.e., when $A_{t-1} = 0$.

We now describe the recursive dynamics of these random variables. After choosing w_1 and before choosing η_1 we have precisely d active vertices hence $\tilde{A}_0 = d$. If $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}$ and the edge (w_t, η_t) was retained in the percolation then we mark $k-1$ neutral vertices as active vertices, and one active vertex as explored, so $A_t = \tilde{A}_{t-1} + (k-1) - 1$. Also, if $[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}$ but the edge (w_t, η_t) was *not* retained in the percolation then $A_t = \tilde{A}_{t-1} - 1$. If $\eta_t \in \tilde{\mathbf{A}}_{t-1}$ then we mark two active vertices as explored and hence $A_t = \tilde{A}_{t-1} - 2$. Together this gives

$$A_t = \tilde{A}_{t-1} + \mathbf{1}_{\{(w_t, \eta_t) \in G_p\}} \sum_{k=2}^d (k-1) \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}\}} - \mathbf{1}_{\{\eta_t \in \tilde{\mathbf{A}}_{t-1}\}} - 1.$$

If $A_t > 0$ then w_{t+1} will be chosen active and so $\tilde{A}_t = A_t$. On the other hand, if $A_t = 0$ then we mark the neutral vertices in $[w_{t+1}]$ (including w_{t+1} itself) as active, and hence $\tilde{A}_t = N(w_{t+1})$. This together with the previous display and (19) gives

$$\begin{aligned}
A_0 &= 0, \quad \tilde{A}_0 = d, \\
A_t &= A_{t-1} + \xi_t + N(w_t). \tag{20}
\end{aligned}$$

Let $0 = t_0 < t_1 < t_2 < \dots$ be the times at which $A_{t_j} = 0$. At time t_j we completely explored the j -th component (which we have started exploring in time $t_{j-1} + 1$) and all the d -tuples that became completely explored between times $t_{j-1} + 1$ and t_j are the vertices of this component. Define the random variables

$$\begin{aligned} S_j &= \left| \left\{ t \in (t_{j-1}, t_j] : (w_t, \eta_t) \in G_p \text{ and } \eta_t \text{ is drawn neutral} \right\} \right|, \\ U_j &= \left| \left\{ t \in (t_{j-1}, t_j] : \eta_t \in \tilde{\mathbf{A}}_{t-1} \right\} \right|, \\ V_j &= \left| \left\{ t \in (t_{j-1}, t_j] : [\eta_t] \in \cup_{i=1}^{d-1} \tilde{\mathbf{N}}_t^{(i)} \right\} \right|. \end{aligned}$$

The following lemma relates all the above to component sizes of the graph $G^*(n, d, p)$.

Lemma 10 *The size of the j -th completely explored component is $S_j + 1$. Furthermore, we have*

$$0 \leq S_j + 1 - \frac{t_j - t_{j-1}}{d-1} \leq \frac{U_j}{d-1} + V_j + 1.$$

Proof. At each time where η_t is neutral and $(w_t, \eta_t) \in G_p$ we add a new d -tuple to our currently explored component, increasing its size by 1. Thus, the size of the j -th completely explored component is simply $S_j + 1$. To get the second part of the lemma denote by T_j the random variable

$$T_j = \left| \left\{ t \in (t_{j-1}, t_j] : (w_t, \eta_t) \notin G_p \text{ and } \eta_t \text{ is drawn neutral} \right\} \right|.$$

Observe that since η_t is drawn among the neutral and active vertices remaining we have

$$t_j - t_{j-1} = S_j + T_j + U_j. \quad (21)$$

Consider now the dynamics described in the two paragraphs preceding (20). By the previous display, and since $A_{t_{j-1}} = A_{t_j} = 0$ we have

$$0 = N(w_{t_{j-1}+1}) - 1 - 2U_j - T_j + \sum_{k=1}^d (k-2) \cdot \left| \left\{ t \in (t_{j-1}, t_j] : (w_t, \eta_t) \in G_p \text{ and } [\eta_t] \in \tilde{\mathbf{N}}_t^{(k)} \right\} \right|.$$

The last sum in the equation can be bounded above by $(d-2)S_j$ and below by $(d-2)S_j - (d-1)V_j$. This together with (21) and the fact that $1 \leq N(w_{t_{j-1}+1}) \leq d$ gives that

$$0 \leq S_j + 1 - \frac{t_j - t_{j-1}}{d-1} \leq \frac{U_j}{d-1} + V_j + 1.$$

□

It will be more convenient to work with the process $\{Y_t\}$ defined by

$$Y_0 = d, \quad Y_t = Y_{t-1} + \xi_t.$$

There is an evident connection between the process $\{Y_t\}$ and $\{A_t\}$. By (20) we have

$$Y_t = A_t - Z_t,$$

where

$$Z_t = \sum_{i=1}^t N(w_i).$$

Observe that Z_t is an increasing process and $Z_t = Z_{t_j+1}$ for all $t \in \{t_j + 1, \dots, t_{j+1}\}$. As $A_{t_j} = 0$ we have that $Y_{t_j} = -Z_{t_j}$ for all j . Thus, for any $t \in \{t_j + 1, \dots, t_{j+1} - 1\}$ we have

$$Y_{t_{j+1}} = -Z_{t_{j+1}} = -Z_t < Y_t,$$

as $A_t > 0$ for such t 's. By induction we learn that $Y_{t_{j+1}} < Y_t$ for all $t < t_{j+1}$. Hence, the t_j 's are record minima for the process $\{Y_t\}$. Since $N(w_{t_j+1}) \leq d$ we have that $Z_{t_{j+1}} \leq -Y_{t_{j+1}} + d$. Thus, by our previous discussion we learn that $Z_t \leq -\min_{s \leq t} Y_s + d$. We conclude that

$$A_t \leq Y_t - \min\{Y_s : s \leq t\} + d. \quad (22)$$

4 Exploration Process Analysis

For the following, we assume that $\varepsilon = \varepsilon(n)$ is a sequence such that $\varepsilon(n) \rightarrow 0$ and we write $p = p(n) = \frac{1+\varepsilon(n)}{d-1}$. Let \mathcal{F}_t be the σ -algebra

$$\mathcal{F}_t = \sigma\left\{N_j^{(k)}, \tilde{N}_j^{(k)} : 0 \leq j \leq t, 0 \leq k \leq d\right\}.$$

At each time t we have that η_t is chosen uniformly among the $dn - 2t + 1$ neutral and active vertices remaining (which are not w_t). Thus for any $0 \leq k \leq d$ we have

$$\mathbf{E}\left[\mathbf{1}_{\{\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(k)}\}} \mid \mathcal{F}_{t-1}\right] = \frac{k\tilde{N}_{t-1}^{(k)}}{dn - 2t + 1}, \quad (23)$$

and

$$\mathbf{E} \left[\mathbf{1}_{\{\eta_t \in \tilde{\mathbf{A}}_{t-1}\}} \mid \mathcal{F}_{t-1} \right] = \frac{\tilde{A}_{t-1}}{dn - 2t + 1}, \quad (24)$$

hence

$$\mathbf{P} \left([\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)} \right) = \frac{k \mathbf{E} \tilde{N}_{t-1}^{(k)}}{dn - 2t + 1}, \quad \mathbf{P} \left(\eta_t \in \tilde{\mathbf{A}}_{t-1} \right) = \frac{\mathbf{E} \tilde{A}_{t-1}}{dn - 2t + 1}. \quad (25)$$

In the conditions of Lemmas 11 - 13 below and Corollary 14 appears a constant C and the constants implicit in the O -notation depend on C .

Lemma 11 *For any $C > 0$ we have that for all $t < C\varepsilon(n)n$*

$$\mathbf{E} A_t = O(\varepsilon t + \sqrt{t}), \quad (26)$$

and

$$\mathbf{E} Z_t = O(\varepsilon t + \sqrt{t}). \quad (27)$$

Lemma 12 *For any $C > 0$ we have that for all $t < C\varepsilon(n)n$*

$$\mathbf{E} \tilde{N}_t^{(d)} = n - t + O(\varepsilon t + \sqrt{t}), \quad (28)$$

$$\mathbf{E} \tilde{N}_t^{(d-1)} = (1-p)t + O(\varepsilon t + \sqrt{t}), \quad (29)$$

$$\mathbf{E} \tilde{N}_t^{(k)} = O(\varepsilon t), \quad 0 < k < d-1, \quad (30)$$

Lemma 13 *For any $C > 0$ we have that for all $t < C\varepsilon(n)n$*

$$\mathbf{E} |\tilde{N}_t^{(k)} - \mathbf{E} \tilde{N}_t^{(k)}| \leq O(\varepsilon t + \sqrt{t}), \quad 0 \leq k \leq d. \quad (31)$$

Corollary 14 *For any $C > 0$ we have that for all $t < C\varepsilon(n)n$*

$$(i) \quad \mathbf{E} \xi_t - \varepsilon + \frac{d-2}{d(d-1)} \cdot \frac{t}{n} = O\left(\varepsilon^2 + \frac{\sqrt{t}}{n}\right),$$

$$(ii) \quad \mathbf{E} \left| \mathbf{E} [\xi_t \mid \mathcal{F}_{t-1}] - \mathbf{E} \xi_t \right| = O\left(\frac{\varepsilon t + \sqrt{t}}{n}\right),$$

$$(iii) \quad \mathbf{E} [\xi_t^2 \mid \mathcal{F}_{t-1}] - (d-2) = O(\varepsilon).$$

Lemma 15 *For any small $\delta > 0$ there exists some constant $c = c(\delta) > 0$ such that if $t \leq \delta n$ then*

$$\mathbf{P}\left(\tilde{N}_t^{(d)} > n - (1 - 3\delta)t\right) \leq e^{-ct}, \quad (32)$$

and

$$\mathbf{P}\left(\tilde{N}_t^{(0)} < pt(1 - 3\delta)\right) \leq e^{-ct}. \quad (33)$$

In order to bound the terms U_j and V_j in Lemma 10 we have the following lemma.

Lemma 16 *For an integer $0 < T < n/4$ define*

$$U_T = \left\{ t \leq T : \eta_t \in \tilde{\mathbf{A}}_{t-1} \text{ or } \eta_t \in \cup_{i=1}^{d-1} \tilde{\mathbf{N}}_{t-1}^{(i)} \right\}.$$

Then there exists some constant $c > 0$ such that if $4\sqrt{n} < T < n/4$

$$\mathbf{P}\left(|U_T| > \frac{4T^2}{n}\right) \leq e^{-cT^2/n}.$$

Proof of Lemma 11. We rely on the inequality (22). It is clear that

$$\sum_{k=2}^d (k-1) \mathbf{1}_{\{\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(k)}\}} \leq d-1, \quad (34)$$

hence (19) implies that $\mathbf{E}[\xi_t | \mathcal{F}_{t-1}] \leq \varepsilon$ and so $\mathbf{E}Y_t = O(\varepsilon t)$ and the process $\{\varepsilon j - Y_j\}_{j \geq 0}$ is a submartingale. Doob's maximal L^2 inequality (see [12]) gives

$$\mathbf{E}[\max_{j \leq t} (\varepsilon j - Y_j)^2] \leq 4\mathbf{E}[(\varepsilon t - Y_t)^2]. \quad (35)$$

By (19) and (23) we have

$$\mathbf{E}[\xi_j | \mathcal{F}_{j-1}] \geq \frac{(1 + \varepsilon)}{d-1} \cdot \frac{d(d-1)\tilde{N}_{j-1}^{(d)} - \tilde{A}_{j-1}}{dn - 2j + 1} - 1.$$

By (14) for all j we have $\tilde{N}_{j-1}^{(d)} \geq n - 2j$ and by (20) we have $\tilde{A}_{j-1} \leq d + (d-2)j$. We deduce by the previous display that $\mathbf{E}[\xi_j | \mathcal{F}_{j-1}] \geq -D\varepsilon$ for some fixed $D > 0$ and all $j < C\varepsilon n$. We learn that for any $k < j < C\varepsilon n$

$$|\mathbf{E}[\xi_j - D\varepsilon | \mathcal{F}_k]| = O(\varepsilon).$$

It follows that for any $k < j$

$$\mathbf{E}[(\xi_j - D\varepsilon)(\xi_k - D\varepsilon)] = O(\varepsilon^2).$$

We deduce from the above that for $t < C\varepsilon n$

$$\mathbf{E}[(\varepsilon t - Y_t)^2] = 2 \sum_{j < k}^t \mathbf{E}[(\xi_j - \varepsilon)(\xi_k - \varepsilon)] + \sum_{j \leq t} \mathbf{E}[(\xi_j - \varepsilon)^2] = O(\varepsilon^2 t^2 + t).$$

By Jensen inequality and (35) we get that

$$\mathbf{E}[\min_{j \leq t}(Y_j - \varepsilon j)] = O(\varepsilon t + \sqrt{t}),$$

and inequality (22) concludes the proof of (26). As $Z_t = A_t - Y_t$, and $\mathbf{E} Y_t = O(\varepsilon t)$ this also concludes the proof of (27). \square

Proof of Lemma 12. As $[w_t] \in \mathbf{N}_{t-1}^{(d)}$ implies that $A_{t-1} = 0$ we have by (14) that

$$\tilde{N}_t^{(d)} \geq \tilde{N}_{t-1}^{(d)} - \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(d)}\}} - \mathbf{1}_{\{A_{t-1}=0\}}.$$

As $\tilde{N}_0^{(d)} = n - 1$ we learn that $\tilde{N}_t^{(d)} \geq n - t - \sum_{i=1}^{t-1} \mathbf{1}_{\{A_i=0\}}$ and so by the definition of Z_t we have that $\tilde{N}_t^{(d)} \geq n - t - Z_t$. Thus (27) of Lemma 11 gives that

$$\mathbf{E} \tilde{N}_t^{(d)} \geq n - t - O(\varepsilon t + \sqrt{t}).$$

Also, by (14) we have that

$$\tilde{N}_t^{(d)} \leq \tilde{N}_{t-1}^{(d)} - \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(d)}\}}.$$

Hence, (23) and $1 - x \leq e^{-x}$ give that

$$\mathbf{E} \left[\tilde{N}_t^{(d)} \mid \mathcal{F}_{t-1} \right] \leq \tilde{N}_{t-1}^{(d)} \left(1 - \frac{d}{dn - 2t + 1} \right) \leq \tilde{N}_{t-1}^{(d)} e^{-\frac{d}{dn - 2t + 1}}.$$

By iterating this we get that

$$\mathbf{E} \tilde{N}_t^{(d)} \leq n e^{-d \sum_{i=0}^t \frac{1}{dn - 2i + 1}} \leq n e^{-\frac{t}{n}} \leq n - t + \frac{t^2}{2n},$$

where the last inequality is due to $e^{-x} \leq 1 - x + x^2/2$ for all $x > 0$. This concludes the proof of (28) as $\frac{t^2}{n} = O(\varepsilon t)$ for $t < C\varepsilon n$.

Observe that (15) and (16) implies that

$$\mathbf{E} \left[\tilde{N}_t^{(d-1)} \mid \mathcal{F}_{t-1} \right] \leq \tilde{N}_{t-1}^{(d-1)} + (1-p),$$

which by iterating yields $\mathbf{E} N_t^{(d-1)} \leq (1-p)t$. To complement this with a lower bound we use (15) and (23) to get

$$\mathbf{E} \left[\tilde{N}_t^{(d-1)} \mid \mathcal{F}_{t-1} \right] \geq \tilde{N}_{t-1}^{(d-1)} + (1-p) \frac{d\tilde{N}_{t-1}^{(d)}}{dn-2t+1} - \frac{(d-1)\tilde{N}_{t-1}^{(d-1)}}{dn-2t+1} - \mathbf{1}_{\{A_{t-1}=0\}}.$$

We now take expectation and bound the second term of the right hand side using (28) and the third term by $\tilde{N}_{t-1}^{(d-1)} \leq t$ for all t . This yields

$$\mathbf{E} \tilde{N}_t^{(d-1)} \geq \mathbf{E} \tilde{N}_{t-1}^{(d-1)} + (1-p) \frac{d(n-t-O(\varepsilon t + \sqrt{t}))}{dn-2t+1} - O(t/n) - \mathbf{P}(A_{t-1}=0).$$

By iterating and using (27) we get

$$\mathbf{E} \left[N_t^{(d-1)} \right] \geq (1-p) \sum_{i=1}^t \left(1 - \frac{(d-2)i+1+O(\varepsilon i + \sqrt{i})}{dn-2i+1} \right) - O(t^2/n) - O(\varepsilon t + \sqrt{t}).$$

The sum can be bounded below by $t - O(t^2/n)$ and as $t^2/n = O(\varepsilon t)$ for $t \leq C\varepsilon n$ we conclude the proof of (29).

To prove the bound (30) note that by (15) we have

$$\mathbf{E} [\tilde{N}_t^{(k)} \mid \mathcal{F}_{t-1}] \leq \tilde{N}_{t-1}^{(k)} + \mathbf{P}([\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k+1)}).$$

As $\tilde{N}_t^{(k+1)} \leq t$ for $k < d-1$, using (25) and iterating gives (30). \square

Proof of Lemma 13. By Lemma 12 and the triangle inequality, the assertion of the lemma is trivial for $k \in \{1, \dots, d-2\}$ as $\frac{t^2}{n} = O(\varepsilon t)$ for $t < C\varepsilon n$. We first prove the assertion for $k=0$. By iterating (17) and (18) we get that

$$\begin{aligned} \tilde{N}_t^{(0)} = 1 &+ \sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \in G_p\}} \mathbf{1}_{\{\eta_i \text{ is neutral}\}} \\ &+ \sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \notin G_p\}} \mathbf{1}_{\{[\eta_i] \in \tilde{\mathbf{N}}_{i-1}^{(1)}\}} + \sum_{i=1}^t \mathbf{1}_{\{w_{i+1} \text{ is neutral}\}}. \end{aligned}$$

Write

$$\begin{aligned} X_1(t) &= \sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \in G_p\}} \mathbf{1}_{\{\eta_i \text{ is neutral}\}}, \\ X_2(t) &= \sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \notin G_p\}} \mathbf{1}_{\{[\eta_i] \in \tilde{\mathbf{N}}_{i-1}^{(1)}\}}, \\ X_3(t) &= \sum_{i=1}^t \mathbf{1}_{\{w_{i+1} \text{ is neutral}\}}. \end{aligned}$$

By definition $X_3(t) \leq Z_{t+1}$, hence the triangle inequality implies

$$\mathbf{E} \left| X_3(t) - \mathbf{E} X_3(t) \right| \leq 2\mathbf{E} Z_{t+1} = O(\varepsilon t + \sqrt{t}), \quad (36)$$

where the last inequality is due to (27). By (25) and (30) we have for $i < C\varepsilon n$

$$\mathbf{E} \left| \mathbf{1}_{\{[\eta_i] \in \mathbf{N}_{i-1}^{(1)}\}} - \mathbf{E} \mathbf{1}_{\{[\eta_i] \in \mathbf{N}_{i-1}^{(1)}\}} \right| \leq \frac{\varepsilon i}{n},$$

and hence the triangle inequality gives that

$$\mathbf{E} \left| X_2(t) - \mathbf{E} X_2(t) \right| \leq O\left(\frac{\varepsilon t^2}{n}\right) = O(\varepsilon t). \quad (37)$$

By writing $\mathbf{1}_{\{\eta_i \text{ is neutral}\}} = 1 - \mathbf{1}_{\{\eta_i \in \tilde{A}_{i-1}\}}$ we get by the triangle inequality

$$\begin{aligned} \mathbf{E} \left| X_1(t) - \mathbf{E} X_1(t) \right| &\leq \mathbf{E} \left| \sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \in G_p\}} - pt \right| \\ &+ \mathbf{E} \left| \sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \in G_p\}} \mathbf{1}_{\{\eta_i \in \tilde{A}_{i-1}\}} - p \sum_{i=1}^t \mathbf{P}(\eta_i \in \tilde{A}_{i-1}) \right|. \end{aligned} \quad (38)$$

Since $\sum_{i=1}^t \mathbf{1}_{\{(w_i, \eta_i) \in G_p\}}$ is distributed as $\text{Bin}(t, p)$, the first expectation on the right hand side of (38) is $O(\sqrt{t})$. By (25) and (26) of Lemma 11 we get for each $i \leq t < C\varepsilon n$,

$$\mathbf{P}(\eta_i \in \tilde{A}_{i-1}) = \frac{\mathbf{E} \tilde{A}_{i-1}}{dn - 2i + 1} \leq O\left(\frac{\varepsilon t + \sqrt{t}}{n}\right).$$

Therefore,

$$\mathbf{E} \left| X_1(t) - \mathbf{E} X_1(t) \right| \leq O(\varepsilon t + \sqrt{t}).$$

This together with (36) and (37) implies that

$$\mathbf{E} |\tilde{N}_t^{(0)} - \mathbf{E} \tilde{N}_t^{(0)}| = O(\varepsilon t + \sqrt{t}).$$

We now prove the assertion of the lemma for $k = d - 1$. After choosing w_{t+1} and before choosing η_{t+1} we have $2t$ explored vertices and \tilde{A}_t active vertices which belong only to d -tuples with at most $d - 1$ neutral vertices in them. Therefore,

$$\tilde{A}_t + 2t = \sum_{k=0}^{d-1} (d - k) \tilde{N}_t^{(k)},$$

and thus

$$\tilde{N}_t^{(d-1)} = \tilde{A}_t + 2t - \sum_{k=0}^{d-2} (d - k) \tilde{N}_t^{(k)}.$$

Hence the triangle inequality implies that,

$$\mathbf{E} |\tilde{N}_t^{(d-1)} - \mathbf{E} \tilde{N}_t^{(d-1)}| \leq \mathbf{E} |\tilde{A}_t - \mathbf{E} \tilde{A}_t| + d \sum_{k=0}^{d-2} \mathbf{E} |\tilde{N}_t^{(k)} - \mathbf{E} \tilde{N}_t^{(k)}|.$$

As we verified the assertion of the lemma for $k \leq d - 2$, by (26) of Lemma 11 we get the lemma for $k = d - 1$. The assertion for $k = d$ follows immediately by the triangle inequality and the fact that

$$\tilde{N}_t^{(d)} = n - \sum_{k=0}^{d-1} \tilde{N}_t^{(k)}.$$

□

Proof of Corollary 14. We simply use (25) to plug into (19) the bounds obtained in Lemma 12. We get

$$\mathbf{E} \xi_t \leq \frac{1 + \varepsilon}{d - 1} \left[\frac{(d - 1)d(n - t)}{dn - 2t + 1} + \frac{(d - 1)(d - 2)(1 - p)t}{dn - 2t + 1} \right] - O\left(\frac{\varepsilon t + \sqrt{t}}{n}\right) - 1.$$

Writing $1 - p = \frac{d-2-\varepsilon}{d-1}$ and expanding the right hand side gives that

$$\mathbf{E} \xi_t - \varepsilon + \frac{d - 2}{d(d - 1)} \cdot \frac{t}{n} = O\left(\frac{\varepsilon t + \sqrt{t}}{n}\right) = O\left(\varepsilon^2 + \frac{\sqrt{t}}{n}\right),$$

as $t \leq C\varepsilon n$. This proves part (i) of the corollary. Part (ii) follows immediately from (23), Lemma 11 and Lemma 13. To prove part (iii), the

bound on $\mathbf{E}[\xi_t^2 \mid \mathcal{F}_{t-1}]$, we square (19) and estimate it using Lemma 12 and Lemma 11. For any $i \neq j$ we have $\mathbf{1}_{\{\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(i)}\}} \mathbf{1}_{\{\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(j)}\}} = 0$, and also $\mathbf{1}_{\{\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(i)}\}} \mathbf{1}_{\{\eta_t \in \tilde{\mathbf{A}}_{t-1}\}} = 0$. So by (19) we have

$$\mathbf{E}[(\xi_t + 1)^2 \mid \mathcal{F}_{t-1}] = \mathbf{P}((w_t, \eta_t) \in G_p) \sum_{k=2}^d (k-1)^2 \mathbf{P}([\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(i)}) - \mathbf{P}(\eta_t \in \tilde{\mathbf{A}}_{t-1}).$$

For any $k < d$, as $\tilde{N}_t^{(k)} \leq t$, by (25) we have $\mathbf{P}([\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}) = O(t/n)$ and as $\tilde{A}_t \leq d + (d-2)t$ we have $\mathbf{P}(\eta_t \in \tilde{\mathbf{A}}_{t-1}) = O(t/n)$. As $\tilde{N}_{t-1}^{(d)} \geq n - 2t$ by (23) we have that $\mathbf{P}([\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(d)} \mid \mathcal{F}_{t-1}) = 1 - O(t/n)$. All this gives that

$$\mathbf{E}[(\xi_t + 1)^2 \mid \mathcal{F}_{t-1}] = p(d-1)^2(1 - O(t/n)) - O(t/n) = d - 1 + O(\varepsilon),$$

and as $\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}] = O(\varepsilon)$ for $t < C\varepsilon n$ we deduce that $\mathbf{E}[\xi_t^2 \mid \mathcal{F}_{t-1}] = d - 2 + O(\varepsilon)$. \square

Proof of Lemma 15. Note that for any $t < \delta n$ we have $\tilde{N}_{t-1}^{(d)} \geq n(1 - 2\delta)$. Thus, for such times $\tilde{N}_t^{(d)}$ can be stochastically bounded above by $n - \sum_{j=1}^t I_j$ where $\{I_j\}$ are i.i.d. Bernoulli random variables receiving 1 with probability $1 - 2\delta$ and 0 with probability 2δ . By Large Deviation (see [3] section A.14) we get (32).

By the same reasoning, for all times $t < \delta n$ the random variable can be stochastically bounded below by $\sum_{i=1}^t J_i$ where $\{J_i\}$ are i.i.d Bernoulli random variables receiving 1 with probability $p(1 - 2\delta)$ and 0 with probability $1 - p(1 - 2\delta)$, which by Large Deviation yields (33). \square

Proof of Lemma 16. We know that $\tilde{N}_{t-1}^{(k)} \leq 2t$ for all $1 \leq k \leq d-1$ and that $A_t \leq d + (d-2)t$ for all t . Thus by (23) for all $t < T < n/4$ we have

$$\mathbf{P}(\eta_t \in \cup_{k=1}^{d-1} \tilde{\mathbf{N}}_{t-1}^{(k)} \cup \tilde{\mathbf{A}}_{t-1} \mid \mathcal{F}_{t-1}) \leq \frac{(3d-4)t + d}{dn - 2t + 1} \leq \frac{4T}{n}.$$

Thus we can stochastically bound $|U_T|$ from above by a random variable distributed as $\text{Bin}(T, q)$, where $q = \frac{4T}{n}$. Thus, standard large deviations bounds, see Corollary A.1.10 of [3], conclude the proof. \square

5 Inside the scaling window

In this Section we prove Theorem 2. We follow the strategy laid out in [21].

Proof of Theorem 2, (3). Let $\varepsilon(n) = \lambda n^{-1/3}$ and $p = \frac{1+\varepsilon(n)}{d-1}$. Let α be a random variable which receives $d-2$ with probability p and -1 with probability $1-p$. Let $\{\alpha_i\}$ be i.i.d. random variables distributed as α and let $\{W_t\}$ be the process defined by $W_t = d + \sum_{i=1}^t \alpha_i$. By (19), we can couple $\{Y_t\}$ and $\{W_t\}$ such that $Y_t \leq W_t$ for all t . Let $h = n^{1/3}$ and define $\gamma = \gamma_h$ by

$$\gamma = \min\{t : W_t = 0 \text{ or } W_t \geq h\}.$$

For any $c > 0$ we have

$$\mathbf{E} e^{-c\alpha} = e^c \left(1 + p(e^{-c(d-1)} - 1)\right),$$

and by expanding both exponentials we get

$$\mathbf{E} e^{-c\alpha} = (1 + c + c^2/2 + \dots) \left[1 + p(-c(d-1) + c^2(d-1)^2/2 - \dots)\right].$$

It is straight forward to check if we set $c = 4\varepsilon$, as long as $\varepsilon > 0$ is small enough, we have $\mathbf{E} e^{-c\alpha} \geq 1$. Similarly, if $\varepsilon < 0$ with $|\varepsilon|$ small enough we have that $\mathbf{E} e^{c\alpha} \leq 1$ for $c = 4\varepsilon$. Thus, if $\lambda > 0$, part (i) of Lemma 7 and $1 - e^{-x} \leq x$ for $x > 0$ implies that

$$\mathbf{P}(W_\gamma > 0) \leq \frac{4d\lambda}{1 - e^{-4\lambda}} n^{-1/3}. \quad (39)$$

A similar computation and an application of part (ii) of Lemma 7 shows that for $\lambda < 0$ and n large enough we have

$$\mathbf{P}(W_\gamma > 0) \leq \frac{-5d\lambda}{e^{-4\lambda} - 1} n^{-1/3}. \quad (40)$$

Also, when $\lambda = 0$ the process $\{W_t\}$ is a martingale and we deduce by optional stopping that $\mathbf{P}(W_\gamma > 0) \leq dn^{-1/3}$. We now estimate $\mathbf{E} \gamma$ for all λ . Assume first $\lambda > 1/4$; as $\{W_t - t\lambda n^{-1/3}\}$ is a martingale, the optional stopping theorem gives

$$d = \mathbf{P}(W_\gamma > 0) \mathbf{E}[W_\gamma \mid W_\gamma > 0] - \lambda n^{-1/3} \mathbf{E} \gamma.$$

We use $1 - e^{-4\lambda} > 1/2$ for $\lambda > 1/4$ in (39) and the fact that $\mathbf{E}[W_\gamma \mid W_\gamma > 0] \leq n^{1/3} + d$ to rearrange the last display. This gives that $\mathbf{E} \gamma \leq 8dn^{1/3}$, for

$\lambda > 1/4$. It is straight forward to check that $\{W_t^2 - \frac{1}{2}t\}$ is a submartingale for any $\lambda > 0$, hence by optional stopping,

$$1 \leq \mathbf{P}(W_\gamma > 0) \mathbf{E}[W_\gamma^2 \mid W_\gamma > 0] - \frac{1}{2} \mathbf{E}\gamma.$$

We use $\frac{4\lambda}{1-e^{-4\lambda}} \leq 2$ for $\lambda \in (0, 1/4]$ in (39) and the obvious estimate $\mathbf{E}[W_\gamma^2 \mid W_\gamma > 0] \leq (n^{1/3} + d)^2$ to rearrange the last display. This gives that $\mathbf{E}\gamma \leq 8dn^{1/3}$, for $\lambda \in (0, 1/4]$. An almost identical computation for the case $\lambda \leq 0$ yields that for all $\lambda \in \mathbb{R}$ we have

$$\mathbf{E}\gamma \leq 8dn^{1/3}.$$

Define $\gamma^* = \gamma \wedge n^{2/3}$; by the last display, inequalities (39) and (40) we deduce that there exists $C = C(\lambda)$ such that

$$\mathbf{P}(W_\gamma^* > 0) \leq \mathbf{P}(W_\gamma \geq n^{1/3}) + \mathbf{P}(\gamma \geq n^{2/3}) \leq Cn^{-1/3}. \quad (41)$$

Taking an exponential in (19) gives

$$\mathbf{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}] = e^{-c} \mathbf{E} \left[e^{c \mathbf{1}_{\{(w_t, \eta_t) \in G_p\}}} \left(\sum_{k=2}^d (k-1) \mathbf{1}_{\{[\eta_t] \in \tilde{\mathbf{N}}_{t-1}^{(k)}\}}^{-1} \mathbf{1}_{\{\eta_t \in \tilde{\mathbf{A}}_{t-1}\}} \right) \mid \mathcal{F}_{t-1} \right].$$

The conditional expectation on the right hand side of the last display is $e^{c(k-1)}$ with probability $\frac{pk\tilde{N}_{t-1}^{(k)}}{dn-2t+1}$ for any $2 \leq k \leq d$ by (23) and at most 1 with probability $1 - p + \frac{p\tilde{A}_{t-1}}{dn-2t+1}$ by 24. Thus,

$$\mathbf{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}] \leq e^{-c} \left[1 + p \left(-1 + \sum_{k=1}^d e^{c(k-1)} \frac{k\tilde{N}_{t-1}^{(k)}}{dn-2t+1} + \frac{\tilde{A}_{t-1}}{dn-2t+1} \right) \right].$$

Using $e^x \leq 1 + x + x^2$ for $x \in [0, 1]$ we have that for $c < \frac{1}{d-1}$

$$\begin{aligned} \mathbf{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}] &\leq e^{-c} \left[1 + p \left(-1 + \sum_{k=1}^d \left(1 + c(k-1) + c^2(k-1)^2 \right) \frac{k\tilde{N}_{t-1}^{(k)}}{dn-2t+1} \right. \right. \\ &\quad \left. \left. + \frac{\tilde{A}_{t-1}}{dn-2t+1} \right) \right]. \end{aligned}$$

We expand the right hand side of the last display using the fact that $\frac{\tilde{A}_{t-1} + \sum_{k=1}^d k\tilde{N}_{t-1}^{(k)}}{dn-2t+1} = 1$ and that $\sum_{k=1}^{d-1} k\tilde{N}_{t-1}^{(k)} \leq (d-1)(n - \tilde{N}_{t-1}^{(d)} - \tilde{N}_{t-1}^{(0)})$. This gives,

$$\begin{aligned} \mathbf{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}] &\leq e^{-c} \left[1 + p \left(\frac{c(d-1)dN_{t-1}^{(d)} + c(d-2)(d-1)(n - N_{t-1}^{(d)} - N_{t-1}^{(0)})}{dn-2t+1} \right. \right. \\ &\quad \left. \left. + c^2(d-1)^2 \right) \right]. \end{aligned} \quad (42)$$

For some small $\delta > 0$ denote by \mathcal{A} the event

$$\mathcal{A} = \{N_t^{(d)} \leq n - (1 - \delta)t, \quad N_t^{(0)} > (1 - \delta)pt, \quad \forall n^{1/3} < t < \delta n/3\}.$$

We now condition on \mathcal{A} and put $p = \frac{1+\varepsilon}{d-1}$ in (42). A straightforward computation yields that for $c < \frac{1}{d-1}$ and $n^{1/3} < t < \delta n/3$ we have

$$\mathbf{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}, \mathcal{A}] \leq e^{-c} \left[1 + (1 + \varepsilon) \left(c - \frac{(d-2+O(\delta))ct}{d(d-1)n} + c^2(d-1) \right) \right].$$

As $d \geq 3$ we can choose δ small enough such that $\frac{(d-2+O(\delta))}{d} > \frac{1}{4}$; we also use $1 + x \leq e^x$ for all $x > 0$ in the last display. This gives that for such t 's,

$$\mathbf{E}[e^{c\xi_t} \mid \mathcal{F}_{t-1}, \mathcal{A}] \leq e^{c\varepsilon - \frac{ct}{4(d-1)n} + 2(d-1)c^2}. \quad (43)$$

By estimating $e^{c\xi_j} \leq e^{c(d-2)}$ for all $j \leq n^{1/3}$, as $\gamma^* \leq n^{2/3}$ we get from the last display that for any $t < \delta n/3 - n^{2/3}$

$$\mathbf{E} \left[e^{c \sum_{j=1}^t \xi_{\gamma^*+j}} \mid \gamma^*, \mathcal{A} \right] \leq e^{c\varepsilon t - \frac{ct^2}{8(d-1)n} + 2(d-1)c^2 t + c(d-2)n^{1/3}}. \quad (44)$$

Define the process $\{R_t\}$ by

$$R_t = Y_{\gamma^*+t} - Y_{\gamma^*} = \sum_{j=1}^t \xi_{\gamma^*+j}.$$

As the estimate (44) is uniform in W_{γ^*} and γ^* we get that

$$\mathbf{E}[e^{cR_t} \mid W_{\gamma^*}, \mathcal{A}] \leq e^{c\varepsilon t - \frac{ct^2}{8(d-1)n} + 2(d-1)c^2 t + c(d-2)n^{1/3}}.$$

Write \mathbf{P}_W for the conditional probability measure given W_{γ^*} and \mathcal{A} . Then by previous equation, for any $c < \frac{1}{d-1}$ and $t < \delta n/3 - n^{2/3}$ we have

$$\begin{aligned} \mathbf{P}_W(R_t \geq -W_{\gamma^*}) &\leq \mathbf{P}_W(e^{cR_t} \geq e^{-cW_{\gamma^*}}) \\ &\leq e^{c\varepsilon t - \frac{ct^2}{8(d-1)n} + 2(d-1)c^2 t + c(d-2)n^{1/3}} e^{cW_{\gamma^*}}. \end{aligned}$$

By (32) and (33) of Lemma 15 it follows that $\mathbf{P}(\mathcal{A}^c) \leq ne^{-an^{1/3}}$ for some fixed $a > 0$. As $Y_{\gamma^*} \leq W_{\gamma^*}$ it follows by the definition of R_t and by conditioning on \mathcal{A} that

$$\begin{aligned} \mathbf{P}(Y_{\gamma^*+t} > 0 \mid W_{\gamma^*} > 0) &\leq \mathbf{E}[\mathbf{P}_W(R_t \geq -W_{\gamma^*}) \mid W_{\gamma^*} > 0] + \mathbf{P}(\mathcal{A}^c) \\ &\leq e^{c\varepsilon t - \frac{ct^2}{8(d-1)n} + 2(d-1)c^2 t + c(d-2)n^{1/3}} \mathbf{E}[e^{cW_{\gamma^*}} \mid W_{\gamma^*} > 0] + ne^{-an^{1/3}}. \end{aligned}$$

Since $W_{\gamma^*} \leq n^{1/3} + d$ we can bound the conditional expectation on the right hand side. This yields,

$$\mathbf{P}\left(Y_{\gamma^*+t} > 0 \mid W_{\gamma^*} > 0\right) \leq e^{c\varepsilon t - \frac{ct^2}{8(d-1)n} + 2(d-1)c^2t + c(d-1)n^{1/3} + cd} + ne^{-an^{1/3}}.$$

Now recall that $\varepsilon = \lambda n^{-1/3}$ and take $c = \frac{\frac{t^2}{8(d-1)n} - \varepsilon t - (d-1)n^{1/3}}{4t(d-1)}$ and $t = Bn^{2/3}$ for some $B > 0$ large enough so that $c > 0$. Note that c is of order $n^{-1/3}$ so clearly $c < \frac{1}{d-1}$. Putting all this together gives that

$$\begin{aligned} \mathbf{P}\left(Y_{\gamma^*+Bn^{2/3}} > 0 \mid W_{\gamma^*} > 0\right) &\leq e^{-\frac{\left(\frac{B^2}{8(d-1)} - \lambda B - (d-1)\right)^2}{8B(d-1)} + O(n^{-1/3})} + ne^{-\alpha n^{2/3}} \\ &\leq e^{-rB^3}, \end{aligned}$$

for some $r = r(\lambda) > 0$ and n large enough. Recall that t_1 is the first time the process Y_t hits 0 and that Lemma 10 implies that $|\mathcal{C}(v)| \leq t_1$. Thus, by our coupling, if $|\mathcal{C}(v)| > An^{2/3}$ then $W_{\gamma^*} > 0$ and $Y_{\gamma^*+(A-1)n^{2/3}} > 0$. Thus by the previous inequality and (41), for $A > 1$ we have

$$\mathbf{P}(|\mathcal{C}(v)| \geq An^{2/3}) \leq Cn^{-1/3}e^{-r(A-1)^3}.$$

Denote by N_T the number of vertices contained in components larger than T . Observe that $|\mathcal{C}_1| \geq T$ implies $N_T \geq T$. So taking $T = An^{2/3}$ gives

$$\mathbf{P}\left(|\mathcal{C}_1| \geq T\right) \leq \mathbf{P}\left(N_T \geq T\right) \leq \frac{\mathbf{E} N_T}{T} \leq \frac{n\mathbf{P}(|\mathcal{C}(v)| \geq T)}{T} \leq \frac{C}{A}e^{-r(A-1)^3},$$

concluding the proof. \square

Proof of Theorem 2, (4). Let $\delta \in (0, 1)$ be small and let $\gamma = \gamma(\delta, \lambda) > 0$ be determined later. Put $h = \gamma n^{1/3}$, $T_1 = n^{2/3}$ and $T_2 = \delta n^{2/3}$. As in [21] we ensure that with high probability the process $\{Y_t\}$ gets to height h before time T_1 , and then stays positive for at least T_2 steps. This ensures by Lemma 10 that $|\mathcal{C}_1| > \frac{\delta n^{2/3}}{d-1}$ with high probability. Indeed, let us define the stopping time

$$\tau_h = \min\{t \leq T_1 : Y_t \geq h\}$$

if this set is nonempty, and $\tau_h = T_1$ otherwise. Observe that $Y_t^2 - Y_{t-1}^2 = \xi_t^2 + 2\xi_t Y_{t-1}$. By Corollary 14, we have $\mathbf{E}[\xi_t^2 \mid \mathcal{F}_{t-1}] = d - 2 + O(n^{-1/3})$

and $\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}] \geq \Omega(n^{-1/3})$ for $t \leq T_1$. Thus, if $Y_{t-1} \leq h$ and γ is small enough, we have that for all $t \leq T_1$,

$$\mathbf{E} \left[Y_t^2 - Y_{t-1}^2 \mid Y_{t-1} \right] \geq \frac{1}{2}.$$

Hence $Y_{t \wedge \tau_h}^2 - (t \wedge \tau_h)/2$ is a submartingale. As $Y_{\tau_h} \leq h + d$ we have that $\mathbf{E} Y_{\tau_h}^2 \leq (h + d)^2 \leq 2h^2$, so by optional stopping we get

$$2h^2 \geq \mathbf{E} Y_{\tau_h}^2 \geq \frac{1}{2} \mathbf{E} \tau_h \geq \frac{T_1}{2} \mathbf{P}(\tau_h = T_1),$$

hence

$$\mathbf{P}(\tau_h = T_1) \leq \frac{4h^2}{T_1}. \quad (45)$$

Write \mathbf{P}_h for conditional probability given the event $\{\tau_h < T_1\}$ and \mathbf{E}_h for conditional expectation given that event and define

$$\tau_0 = \min\{t \leq T_2 : Y_{\tau_h+t} = 0\}.$$

We wish to bound from above the probability that $\tau_0 \leq T_2$ given that $\tau_h < T_1$. As before there exists a constant $C = C(\lambda)$ such that $\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}] \geq -Cn^{-1/3}$ for all $t \leq T_1 + T_2$. Thus the process

$$S_t = Y_{\tau_h+t} - Y_{\tau_h} + tCn^{-1/3}$$

is a submartingale and hence so is S_t^2 . We conclude that as long as $h > T_2Cn^{-1/3}$

$$\begin{aligned} \mathbf{P}_h \left(\min_{t \leq T_2} Y_{\tau_h+t} \leq 0 \right) &\leq \mathbf{P}_h \left(\min_{t \leq T_2} S_t \leq -h + T_2Cn^{-1/3} \right) \\ &\leq \mathbf{P}_h \left(\max_{t \leq T_2} S_t^2 > (h - T_2Cn^{-1/3})^2 \right) \\ &\leq \frac{4\mathbf{E}_h S_{T_2}^2}{(h - T_2Cn^{-1/3})^2}, \end{aligned} \quad (46)$$

where the last inequality is Doob's Maximal inequality (see [12]). As usual, for any $k < j < T_1 + T_2$ we can bound

$$\mathbf{E}[\xi_j - Cn^{-1/3} \mid \mathcal{F}_{k-1}] = O(n^{-1/3}),$$

and so

$$\mathbf{E}[(\xi_j - Cn^{-1/3})(\xi_k - Cn^{-1/3})] = O(n^{-2/3}).$$

This together with the fact that $\xi_{\tau_h+j} - Cn^{-1/3}$ is bounded by $d-2$ shows that

$$\begin{aligned} \mathbf{E}_h[S_{T_2}^2 \mid \tau_h] &= \sum_{j \neq k}^{T_2} \mathbf{E}_h[(\xi_{\tau_h+j} - Cn^{-1/3})(\xi_{\tau_h+k} - Cn^{-1/3}) \mid \tau_h] \\ &+ \sum_{j=1}^{T_2} \mathbf{E}_h[(\xi_{\tau_h+j} - Cn^{-1/3})^2 \mid \tau_h] = O(n^{-2/3}T_2^2 + T_2) = O(\delta n^{2/3}). \end{aligned}$$

Hence (46) implies that

$$\mathbf{P}_h(\tau_0 \leq T_2) \leq \frac{O(\delta n^{2/3})}{(h - \delta C n^{1/3})^2},$$

as long as $\gamma > \delta C$, so the denominator is positive. Combining this with (45) gives

$$\begin{aligned} \mathbf{P}(\tau_0 \leq T_2) &\leq \mathbf{P}(\tau_h = T_1) + \mathbf{P}_h(\tau_0 \leq T_2) \leq \frac{4h^2}{T_1} + \frac{O(\delta n^{2/3})}{(h - \delta C n^{1/3})^2} \\ &= 4\gamma^2 + \frac{O(\delta)}{(\gamma - \delta C)^2}, \end{aligned}$$

and by choosing $\gamma = \delta C + \delta^{1/4}$ we deduce that

$$\mathbf{P}(\tau_0 \leq T_2) \leq D\delta^{1/2},$$

for some constant $D = D(\lambda) > 0$. By Lemma 10, $|\mathcal{C}_1| \leq \frac{T_2}{d-1}$ implies $\tau_0 \leq T_2$, which concludes the proof. \square

6 Below the scaling window

We use Lemma 8 on another specific case. Fix some small $\varepsilon > 0$ and set $p = \frac{1-\varepsilon}{d-1}$. Let β be a random variable receiving $d-2$ with probability p and -1 with probability $1-p$. Let $\{W_t\}$ and τ be defined as in Lemma 8 with $W_0 = d$.

Lemma 17 *There exists constant $c_1, c_2 > 0$ such that for all $T > \varepsilon^{-2}$ we have*

$$\mathbf{P}(\tau \geq T) \geq c_1 \left(\varepsilon^{-2} T^{-3/2} e^{-\frac{(\varepsilon^2 + c_2 \varepsilon^3)T}{2(d-2)}} \right).$$

Furthermore,

$$\mathbf{E} \tau^2 = O(\varepsilon^{-3}).$$

Proof. We estimate θ_0 defined in Lemma 8. By (8) we have

$$\varphi(\theta_0)^{-1} \left[\frac{e^{\theta_0(d-2)}(d-2)(1-\varepsilon)}{d-1} - \frac{e^{-\theta_0}(d-2+\varepsilon)}{d-1} \right] = 0.$$

By estimating $e^x = 1 + x + O(x^2)$ we get

$$\theta_0 = \frac{\varepsilon}{d-2} + O(\varepsilon^2).$$

We have

$$\varphi(\theta) = \frac{1-\varepsilon}{d-1} e^{\theta(d-2)} + \frac{d-2+\varepsilon}{d-1} e^{-\theta}.$$

Plugging in the value of θ_0 and writing $e^x = 1 + x + \frac{x^2}{2} + O(x^3)$ gives that

$$\varphi(\theta_0) = 1 - \frac{\varepsilon^2}{2(d-2)} + O(\varepsilon^3).$$

Thus by Lemma 8 we have

$$\mathbf{P}(\tau \geq T) = \sum_{\ell \geq T} \Theta \left(\ell^{-3/2} \varphi(\theta_0)^\ell \right).$$

Using our estimate on $\varphi(\theta_0)$ and the assumption that $T > \varepsilon^{-2}$ an immediate computation yields the first assertion of the lemma. The second assertion follows from the following computation. By Lemma 8 we have

$$\mathbf{E} \tau^2 = \sum_{\ell \geq 1} \ell^2 \mathbf{P}(\tau = \ell) \leq C \sum_{\ell \geq 1} \sqrt{\ell} \varphi(\theta_0)^\ell.$$

Thus, by direct computation (or by [14], section XIII.5, Theorem 5)

$$\mathbf{E} \tau^2 \leq O \left(\frac{1}{1 - \varphi(\theta_0)} \right)^{3/2} = O(\varepsilon^{-3}).$$

□

Proof of Theorem 3. Note that Proposition 1 proves the upper bound on $|\mathcal{C}_\ell|$ implied in Theorem 3, so we only need to prove the lower bound. Write

$$T = 2(1 - \eta)(d - 2)\varepsilon^{-2} \log(n\varepsilon^3).$$

For each integer $j \geq 0$ let $\{W_t^{(j)}\}$ be independent processes defined by $W_0^{(j)} = Y_{t_j}$ and $W_t^{(j)} - W_{t-1}^{(j)}$ receives $d - 2$ with probability $\frac{1 - (1 + \frac{\eta}{4})\varepsilon}{d-1}$ and

-1 otherwise. Note that for each j , the process $\{W_t^{(j)}\}$ is just the process defined in Lemma 17 with $p = \frac{1-(1+\frac{\eta}{4})\varepsilon}{d-1}$. By (19) and (23), the variable ξ_t can always be stochastically bounded below by a variable taking the value $d-2$ with probability $\frac{1-\varepsilon}{d-1} \cdot \frac{d\tilde{N}_{t-1}^{(d)}}{dn}$ and -1 otherwise. Since $\tilde{N}_t^{(d)} \geq n-2t$ for all t , as long as $t < \frac{\eta}{8}\varepsilon n$ we can stochastically bound ξ_t below by $W_t^{(j)} - W_{t-1}^{(j)}$. Thus, as long as $t_{j+1} < \frac{\eta}{8}\varepsilon n$, we can couple $\{Y_t\}$ and $W_t^{(j)}$ such that $Y_{t_j+t} \geq W_t^{(j)}$ for all $t \in [0, t_{j+1} - t_j]$. Define the stopping times $\{\tau_j\}$ by

$$\tau_j = \min\{t : W_t^{(j)} = W_0^{(j)} - 1\}.$$

By our coupling, it is clear that if $\tau_j > T$ then $t_{j+1} - t_j > T$. Take

$$N = \left\lfloor \varepsilon^{-1} (n\varepsilon^3)^{(1-\frac{\eta}{8})} \right\rfloor.$$

We will prove that with high probability $t_N < \frac{\eta}{8}\varepsilon n$ and that there exists $k_1 < k_2 < \dots < k_\ell < N$ such that $\tau_{k_i} > T$. Since $\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}] \leq -\varepsilon$, we have by optional stopping that $\mathbf{E}[t_{j+1} - t_j] \leq d\varepsilon^{-1}$, and hence $\mathbf{E}t_N \leq d\varepsilon^{-2}(n\varepsilon^3)^{(1-\frac{\eta}{8})}$ which implies that

$$\mathbf{P}\left(t_N > \frac{\eta}{8}\varepsilon n\right) \leq \frac{8(n\varepsilon^3)^{-\frac{\eta}{8}}}{\eta} \rightarrow 0. \quad (47)$$

Also, by Lemma 17 we have for some $c > 0$

$$\mathbf{P}(\tau_j > T) \geq c\varepsilon(n\varepsilon^3)^{-(1+\frac{\eta}{4})^2(1-\eta)(1-c_2\varepsilon)} \log(n\varepsilon^3)^{-3/2} \geq \varepsilon(n\varepsilon^3)^{-(1-\frac{\eta}{4})},$$

as long as $\eta < 4$ and ε is small enough. Let X be the number of $j \leq N$ such that $\tau_j > T$. Then we have

$$\mathbf{E}X \geq N\varepsilon(n\varepsilon^3)^{-(1-\frac{\eta}{4})} \geq C(n\varepsilon^3)^{\frac{\eta}{8}} \rightarrow \infty,$$

hence by Large Deviations (see [3], section A.14) for any fixed integer $\ell > 0$ we have for some $c > 0$

$$\mathbf{P}\left(X < \ell\right) \leq e^{-c(n\varepsilon^3)^{\frac{\eta}{8}}} \rightarrow 0. \quad (48)$$

Our coupling and Lemma 10 imply that

$$\left\{|\mathcal{C}_\ell| < \frac{T}{d-1}\right\} \subset \left\{X < \ell\right\} \cup \left\{t_N > \frac{\eta}{8}\varepsilon n\right\},$$

and hence by (47) and (48) we have

$$\mathbf{P}\left(|\mathcal{C}_\ell| < \frac{T}{d-1}\right) \rightarrow 0.$$

□

7 Above the scaling window

We split the proof of Theorem 4 into two steps. In the first step we show there is a unique component of order $\frac{2d}{d-2}\varepsilon n$ which has about $2d\varepsilon n$ closed edges separating it from its boundary. In the second step we condition on this event and restart the exploration process on the graph remaining after removing this partial matching to get the estimates on the ℓ -th largest component for $\ell \geq 2$. The first step follows the strategy laid out in [22].

We require some definitions. Consider p -bond percolation on the configuration model, i.e., we draw a perfect matching on the vertex set $\{v_{i,k} : 1 \leq i \leq n, 1 \leq k \leq d\}$ and then retain each edge with probability p and delete it with probability $1 - p$ independently of all other edges. Denote the resulting graph by $M(n, d, p)$ and recall that $G^*(n, d, p)$ is the graph obtained from $M(n, d, p)$ by contracting every tuple to a vertex. A set of d -tuples S in $M(n, d, p)$ is called a *component* if the vertex set corresponding to S in $G^*(n, d, p)$ is a connected component. We say that a d -tuple v is *k -damaged*, with $0 \leq k \leq d$, by a component S if $v \notin S$ and there are precisely k closed edges (i.e., edges not retained in percolation) between a vertex in v and a vertex in a tuple belonging to S . Let $M_k(S)$ be the set of all k -damaged tuples of a component S . Let $p = \frac{1+\varepsilon(n)}{d-1}$. We say a component S is *δ -giant* for some $\delta > 0$ if the following properties hold:

- (i) $(1 - \delta)\frac{2d}{d-2}\varepsilon n \leq |S| \leq (1 + \delta)\frac{2d}{d-2}\varepsilon n$,
- (ii) $(1 - \delta)2d\varepsilon n \leq |M_1(S)| \leq (1 + \delta)2d\varepsilon n$.

For $\delta > 0$ let $\mathcal{G}(\delta)$ denote the event that there exists a unique δ -giant component in $M(n, d, p)$. The following Theorems imply Theorem 4.

Theorem 18 *Let $\varepsilon(n) > 0$ be a sequence such that $\varepsilon(n) \rightarrow 0$ and $\varepsilon(n)n^{1/3} \rightarrow \infty$. Let $p = \frac{1+\varepsilon(n)}{d-1}$ and consider $M(n, d, p)$. Then for any $\delta > 0$ we have*

$$\mathbf{P}(\mathcal{G}(\delta)) \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (49)$$

Theorem 19 *Condition on $\mathcal{G}(\delta)$ and denote by S_1 the δ -giant component. Let $\{S_\ell\}_{\ell \geq 2}$ denote the components of $M(n, d, p)$ after removing S_1 , ordered by size. Then under the conditions of the previous theorem, for any $\eta > 0$ there is $\delta > 0$ small enough such that*

$$\mathbf{P}\left(|S_2| \geq (1 + \eta)\frac{2(d-2)}{d-1}\varepsilon^{-2}(n) \log(n\varepsilon^3(n)) \mid \mathcal{G}(\delta)\right) \rightarrow 0, \quad (50)$$

and for any fixed integer $\ell \geq 2$ we have

$$\mathbf{P}\left(|S_\ell| \leq (1 - \eta) \frac{2(d-2)}{d-1} \varepsilon^{-2}(n) \log(n\varepsilon^3(n)) \mid \mathcal{G}(\delta)\right) \rightarrow 0. \quad (51)$$

Proof of Theorem 4. Fix some $\eta > 0$ and take $\delta > 0$ small enough guaranteed by Theorem 19. Theorem 18 guarantees that the event $\mathcal{G}(\delta)$ holds with high probability. Hence with that probability and hence there exists a component of size between $(1 - \delta) \frac{2d}{d-2} \varepsilon n$ and $(1 + \delta) \frac{2d}{d-2} \varepsilon n$. We condition on $\mathcal{G}(\delta)$ and remove this component; Theorem 19 then implies that with high probability the graph remaining has no components of size bigger than $(1 + \eta) \frac{2(d-2)}{d-1} \varepsilon^{-2}(n) \log(n\varepsilon^3(n))$ and the ℓ -th component is bigger than $(1 - \eta) \frac{2(d-2)}{d-1} \varepsilon^{-2}(n) \log(n\varepsilon^3(n))$. As these probabilities tend to 0 in the space $G^*(n, d, p)$, as the event *Simple* has positive probability, we conclude the same for the space $G(n, d, p)$. \square

Proof of Theorem 18. Write $T = \frac{(1+\delta)2d(d-1)}{d-2} \varepsilon n$ and $\xi_j^* = \mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}]$. The process

$$M_t = Y_t - \sum_{j=1}^t \xi_j^*,$$

is a martingale. By Doob's maximal L^2 inequality (see [12]) we have

$$\mathbf{E}(\max_{t \leq T} M_t)^2 \leq 4\mathbf{E} M_T^2.$$

As M_t has orthogonal bounded increments we conclude $\mathbf{E} M_T^2 = O(T)$. By Jensen inequality

$$\mathbf{E} \left[\max_{t \leq T} \left(Y_t - \sum_{j=1}^t \xi_j^* \right) \right] \leq O(\sqrt{T}) = O(\sqrt{\varepsilon n}). \quad (52)$$

By (19) and (23) for any $j \leq T$ we have

$$\xi_j^* - \mathbf{E} \xi_j = \frac{1 + \varepsilon}{d-1} \sum_{k=2}^d (i-1) \left[\frac{k \tilde{N}_{j-1}^{(k)} - i \mathbf{E} \tilde{N}_{j-1}^{(k)}}{dn - 2j + 1} \right] - \frac{\tilde{A}_{j-1} - \mathbf{E} \tilde{A}_{j-1}}{dn - 2j + 1}.$$

Applying the triangle inequality to the last display, together with (26) of Lemma 11 and Lemma 13 gives that $\mathbf{E} |\xi_j^* - \mathbf{E} \xi_j| \leq O\left(\frac{\varepsilon j + \sqrt{j}}{n}\right)$. So for any

$t \leq T$ we have

$$\mathbf{E} \left[\sum_{j=1}^t |\xi_j^* - \mathbf{E} \xi_j| \right] = O(\varepsilon^3 n).$$

By the triangle inequality we get

$$\mathbf{E} \left[\max_{t \leq T} \left| \sum_{j=1}^t (\xi_j^* - \mathbf{E} \xi_j) \right| \right] \leq O(\varepsilon^3 n). \quad (53)$$

Using the triangle inequality together with (52), (53) and Markov's inequality gives

$$\mathbf{P} \left(\max_{t \leq T} \left| Y_t - \sum_{j=1}^t \mathbf{E} \xi_j \right| \geq \delta \varepsilon^2 n \right) \leq \delta^{-1} (O(\varepsilon) + O((\varepsilon^3 n)^{-1/2})) \longrightarrow 0. \quad (54)$$

By Corollary 14 we have that for any $b > 0$

$$\sum_{t=1}^{b\varepsilon n} \mathbf{E} \xi_t = \left(b - \frac{(d-2)b^2}{2d(d-1)} \right) \varepsilon^2 n + O(\varepsilon^3 n). \quad (55)$$

Write

$$t' = \frac{2d(d-1)}{(d-2)} \varepsilon n.$$

Inequalities (54) and (55) imply that for small $\delta > 0$ with probability tending to 1, we have that Y_t is positive at times $[\delta t'/2, t'(1-\delta/2)]$. This together with Lemma 10 implies that with high probability we have explored a component containing at least $(1-\delta) \frac{2d}{d-2} \varepsilon n$ tuples. Furthermore, by (54) and (55) we infer that

$$\mathbf{P} \left(Y_{t'(1+\delta)} \leq -\frac{2d(d-1)}{d-2} \delta (1+\delta) \varepsilon^2 n + O(\varepsilon^3 n) \right) \rightarrow 0,$$

and

$$\mathbf{P} \left(\forall t \leq \delta t'/2 \quad Y_t > O(-\varepsilon^3 n) \right) \rightarrow 1.$$

Thus, with high probability, by time $t'(1+\delta)$ we have completely explored a component of size at least $(1-\delta) \frac{2d}{d-2} \varepsilon n$. On the other hand, Lemma 16 and Lemma 10 show that with high probability the size of this component is at most $(1+\delta) \frac{2d}{d-2} \varepsilon n$. Denote this component by S . By (29) of Lemma 12 and Lemma 13 we have that with high probability $|\tilde{N}_t^{(d-1)} - 2d\varepsilon n| \leq \delta\varepsilon n$. This implies that $|M_1(S) - 2d\varepsilon n| \leq \delta\varepsilon n$ with high probability and concludes our

proof. \square

To prove Theorem 19 we need the following lemma, which is just another application of Lemma 8 to a specific case. Fix some small $\varepsilon > 0$ and let β be a random variable taking the value $d - 2$ with probability $\frac{1-(2d-3)\varepsilon}{d-1}$, the value $d - 3$ with probability 2ε and the value -1 with probability $\frac{d-2-\varepsilon}{d-1}$. Let $\{W_t\}$ and τ be defined as in Lemma 8 with $W_0 = d$.

Lemma 20 *There exists constant $C_1, C_2, c_1, c_2 > 0$ such that for all $T > \varepsilon^{-2}$ we have*

$$\mathbf{P}(\tau \geq T) \leq C_1 \left(\varepsilon^{-2} T^{-3/2} e^{-\frac{(\varepsilon^2 - c_1 \varepsilon^3)T}{2(d-2)}} \right),$$

and

$$\mathbf{P}(\tau \geq T) \geq c_1 \left(\varepsilon^{-2} T^{-3/2} e^{-\frac{(\varepsilon^2 + c_2 \varepsilon^3)T}{2(d-2)}} \right).$$

Furthermore,

$$\mathbf{E} \tau^2 = O(\varepsilon^{-3}).$$

Proof. We estimate θ_0 of Lemma 8. By (8) we have

$$\varphi(\theta_0)^{-1} \left[\frac{e^{\theta(d-2)}(d-2)(1-(2d-3)\varepsilon)}{d-1} + e^{\theta_0(d-3)}(d-3)2\varepsilon - \frac{e^{-\theta_0}(d-2-\varepsilon)}{d-1} \right] = 0.$$

By estimating $e^x = 1 + x + O(x^2)$ we get

$$\theta_0 = \frac{\varepsilon}{d-2} + O(\varepsilon^2).$$

We have

$$\varphi(\theta) = \frac{1-(2d-3)\varepsilon}{d-1} e^{\theta(d-2)} + 2\varepsilon e^{\theta_0(d-3)} + \frac{d-2-\varepsilon}{d-1} e^{-\theta}.$$

By estimating $e^x = 1 + x + x^2/2 + O(x^3)$ and plugging in the value of θ_0 , we obtain that

$$\varphi(\theta_0) = 1 - \frac{\varepsilon^2}{2(d-2)} + O(\varepsilon^3).$$

The rest of the proof is identical to the proof of Lemma 17. \square

Proof of Theorem 19. Let S be the component specified in the event $\mathcal{G}(\delta)$. Condition on $\mathcal{G}(\delta)$ and consider the graph remaining after removing S . Denote by \mathbf{P}_S denote the distribution of this remaining graph conditioned on S and on the edges in the matching adjacent to vertices in the tuples

of S . Denote by \mathbf{P}_M the distribution of p -bond percolation on a uniform matching on a set of $\sum_{k=1}^d M_k(S)$ tuples of which precisely $M_k(S)$ tuples are of size $d - k$. Observe that \mathbf{P}_S is just \mathbf{P}_M conditioned on the event that the resulting graph has no δ -giant component. Theorem 18 guarantees that with high probability there is a unique δ -giant component. We learn that for any set of graphs \mathcal{B} which do not contain an δ -giant component we have $\mathbf{P}_S(\mathcal{B}) = (1 + o(1))\mathbf{P}_M(\mathcal{B})$. Thus it suffices to prove the required tail bounds on the components in \mathbf{P}_M . We do this in a similar manner to the proof Theorem 3.

Given S , the exploration process on the remaining graph, starting from a tuple v has the same dynamics described in Section 3. As S is a δ -giant component, we start this exploration process with $n - (1 + O(\delta))\frac{2d}{d-2}\varepsilon n$ tuples of which $n - (1 + O(\delta))\frac{2d(d-1)}{d-2}\varepsilon n$ are d -tuples and $(1 + O(\delta))2d\varepsilon n$ are $(d-1)$ -tuples. The number of vertices is therefore $dn - (1 + O(\delta))\frac{4d(d-1)}{d-2}\varepsilon n$. In the notation of Section 3 we have

$$\left| \tilde{N}_0^{(d)} - \left(n - \frac{2d(d-1)}{d-2}\varepsilon n \right) \right| \leq \delta\varepsilon n, \quad \left| \tilde{N}_0^{(d-1)} - 2d\varepsilon n \right| \leq \delta\varepsilon n.$$

Fix

$$T = (1 + \eta)2(d-2)\varepsilon^{-2} \log(\varepsilon^3 n).$$

As $|\tilde{N}_t^{(k)} - \tilde{N}_{t-1}^{(k)}| \leq 2$ for every t and k , and $T \leq \delta\varepsilon n$ we learn from (23) that for all $t \leq T$ we have

$$\begin{aligned} \mathbf{P}_M\left(\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(d)} \mid \mathcal{F}_{t-1}\right) &\leq \frac{d\left(n - (1 + O(\delta))\frac{2d(d-1)}{d-2}\varepsilon n\right)}{dn - (1 + O(\delta))\frac{4d(d-1)}{d-2}\varepsilon n} \\ &\leq 1 - (1 + O(\delta))2(d-1)\varepsilon, \end{aligned} \quad (56)$$

$$\begin{aligned} \mathbf{P}_M\left(\eta_t \in \tilde{\mathbf{N}}_{t-1}^{(d-1)} \mid \mathcal{F}_{t-1}\right) &\leq \frac{(d-1)(1 + O(\delta))2d\varepsilon n}{dn - (1 + O(\delta))\frac{4d(d-1)}{d-2}\varepsilon n} \\ &\leq (1 + O(\delta))2(d-1)\varepsilon. \end{aligned} \quad (57)$$

By (19) we can bound $\mathbf{P}_M(\xi_t = d-2 \mid \mathcal{F}_{t-1})$ above by multiplying the right hand side of (56) times p . Similarly, we can bound $\mathbf{P}_M(\xi_t = d-3 \mid \mathcal{F}_{t-1})$ above by multiplying the right hand side of (57) times p . Therefore, we can stochastically bound from above ξ_t by a random variable β taking the value $d-2$ with probability $\frac{1-(1+O(\delta))(2d-3)\varepsilon}{d-1}$, the value $d-3$ with probability

$(1 + O(\delta))2\varepsilon$ and otherwise the value -1 . Recall that t_1 denotes the first hitting time of 0 by the process $\{Y_t\}$. Lemma 20 then gives

$$\mathbf{P}_M(t_1 > T) \leq \varepsilon(n\varepsilon^3)^{-(1+\eta)(1-O(\delta))(1-O(\varepsilon))} \leq \varepsilon(n\varepsilon^3)^{-(1+\eta/2)},$$

as long as δ is small enough. Applying Lemma 16 and Lemma 10 gives that

$$\mathbf{P}_M(|\mathcal{C}(v)| > (1 + \eta)\frac{2(d-2)}{d-1}\varepsilon^{-2}\log(\varepsilon^3 n)) \leq \varepsilon(n\varepsilon^3)^{-(1+\eta/2)},$$

and as in the proof of Proposition 1 this yields that

$$\mathbf{P}_M(|\mathcal{S}_2| > (1 + \eta)\frac{2(d-2)}{d-1}\varepsilon^{-2}\log(\varepsilon^3 n)) \leq (\varepsilon^3 n)^{-\eta/2} \rightarrow 0.$$

The proof that for every fixed $\ell \geq 2$

$$\mathbf{P}_M(|\mathcal{S}_\ell| < (1 - \eta)\frac{2(d-2)}{d-1}\varepsilon^{-2}\log(\varepsilon^3 n)) \rightarrow 0,$$

goes by bounding the process Y_t from below by a process with independent increments. This is carried out almost identically to the proof of Theorem 3 and we omit the details. \square

8 The limiting distribution

Recall the definitions of the processes $B^\lambda(\cdot)$ and $W^\lambda(\cdot)$ in (6) and (7). Throughout this section for a process $\{S_t\}$ indexed by positive integers we write S_t for $t \in \mathbb{R}$ to denote the continuous linear interpolation of S_t .

Recall that $0 = t_0 < t_1 < t_2 < \dots$ are the times at which $A_{t_j} = 0$. Using the process Y_t we define the process \hat{Y}_t by $\hat{Y}_0 = Y_0 = d$ and for any $t \in [t_j, t_{j+1})$

$$\hat{Y}_t = \begin{cases} Y_t, & \text{if } Y_t \geq Y_{t_j}, \\ Y_{t_j} & \text{otherwise,} \end{cases} \quad (58)$$

and $\hat{Y}_t = \hat{Y}_{dn/2}$ for any $t \geq dn/2$. In this manner, the times $\{t_j\}$ are all the record minima of the process $\{\hat{Y}_t\}$. The main theorem of this Section is the following:

Theorem 21 *Fix $\lambda \in \mathbb{R}$ and let $p = \frac{1+\lambda n^{-1/3}}{d-1}$. Then as $n \rightarrow \infty$ we have that*

$$n^{-1/3}\hat{Y}_{((d-1)n^{2/3}\cdot)} \xrightarrow{d} (d-1)B^\lambda(\cdot),$$

where this convergence is on finite intervals.

Theorem 21 states that $n^{-1/3}\widehat{Y}_{((d-1)n^{2/3}, \cdot)}$ converges to the process $(d-1)B^\lambda$. It is thus natural to expect that ordered excursions lengths of $\widehat{Y}_{((d-1)n^{2/3}, \cdot)}$ above past minima, will converge to excursions lengths of B^λ above its past minima. Theorem 5 essentially follows from this assertion, but proving it requires some technical work and we provide the details below. We postpone the proof Theorem 21 to the end of this section.

Fix some $s > 0$ and let $C[0, s]$ be the space of continuous real functions on $[0, s]$. Let $f \in C[0, s]$ and consider the set

$$\mathcal{E} = \{(r, \ell) \subset [0, s] : f(r) = f(\ell) = \min_{u \leq \ell} f(u) \text{ and } f(x) > f(r) \quad \forall r < x < \ell\}.$$

This set defines excursions of f above its past minima. To each excursion (r, ℓ) we associate the length $\ell - r$. Since the sum of excursion lengths is at most s , it is possible to order them in a decreasing order $(\mathcal{L}_1, \mathcal{L}_2, \dots)$. We call a point ℓ , such that $(r, \ell) \in \mathcal{E}$, an excursion *ending* point. We say a function $f \in C[0, s]$ *good* if none of its excursion ending points are local minima and if almost every point in $[0, s]$ is contained in some excursion, i.e. for almost every $x \in [0, s]$ there exists $(r, \ell) \in \mathcal{E}$ such that $r < x < \ell$. Given an integer m , consider the function $\phi_m : C[0, s] \rightarrow \mathbb{R}^m$ defined by

$$\phi_m(f) = (\mathcal{L}_1, \dots, \mathcal{L}_m).$$

Proposition 22 *If $f \in C[0, s]$ is good, then ϕ is continuous at f with respect to the $\|\cdot\|_\infty$ norm.*

Proof. We prove for the case $m = 1$. The proof for $m > 1$ is similar and we omit it. Let $f_n \in C[0, s]$ be a sequence of functions such that $f_n \rightarrow f$. Consider the longest excursion (r, ℓ) such that $\ell - r = \mathcal{L}_1 = \phi_1(f)$. As for any $\varepsilon > 0$ small enough there exists $\delta > 0$ such that $f(x) > f(r) + \delta$ for $x \in (r + \varepsilon, \ell - \varepsilon)$ we conclude that $\liminf_{n \rightarrow \infty} \phi_1(f_n) \geq \phi_1(f)$. On the other hand, as almost every point in $[0, s]$ is inside some excursion of f , for any $\varepsilon > 0$ we can find excursions ending points ℓ_1, \dots, ℓ_k of f such that $\ell_1 \leq \mathcal{L}_1 + \varepsilon$, $s - \ell_k < \mathcal{L}_1 + \varepsilon$ and $\ell_i - \ell_{i-1} < \mathcal{L}_1 + \varepsilon$ for $1 < i \leq k$. Since f is good, for any $\varepsilon > 0$ small enough we can find $\delta > 0$ such that there exists $x_i \in (\ell_i, \ell_i + \varepsilon)$ such that $f(\ell_i) - f(x_i) > \delta$ for all $i \leq k$. It follows that for large enough n , the function f_n has excursion ending points in the intervals $(\ell_i, \ell_i + \varepsilon)$. We conclude that $\limsup_{n \rightarrow \infty} \phi_1(f_n) \leq \phi_1(f)$. \square

Proof of Theorem 5. See [20] or [24] for general background on Brownian Motion and for the proofs of the theorems we use in the following. Fix

some $s > 0$. It is a classic fact that the zero set of Brownian motion has no isolated points and is of 0 measure with probability 1. Also, by a Theorem of Levy we know that $\{B(t) - \min_{y \leq t} B(y)\}_t$ is distributed as $\{|B(t)|\}_t$, so we deduce that with probability 1, a Brownian motion sample path is good. By the Cameron-Martin Theorem, with probability 1 the process $B^\lambda(\cdot)$ is good. As ϕ_m is continuous on almost every sample point of B^λ , and $\phi_m((d-1)B^\lambda) = \phi_m(B^\lambda)$ we deduce by Theorem 21 and Theorem 2.2.3 from [12] that for any integer $m > 0$

$$((d-1)n^{2/3})^{-1} \phi_m(\widehat{Y}_u) \xrightarrow{d} \phi_m(B^\lambda).$$

In Section 3 we showed that the times t_j are record minima of Y_t . Hence, by (58), the lengths $\{t_{j+1} - t_j\}$ are excursions lengths of \widehat{Y}_u above its past minima. Lemma 16 allows us to deduce immediately that for any $s > 0$,

$$n^{-2/3} \left| \left\{ t \leq sn^{2/3} : \eta_t \in \widetilde{\mathbf{A}}_{t-1} \text{ or } [\eta_t] \in \cup_{i=0}^{d-2} \widetilde{\mathbf{N}}_{t-1}^{(i)} \right\} \right| \xrightarrow{d} 0. \quad (59)$$

Thus, if $t_{j+1} - t_j$ is the ℓ -th largest excursion ending before time $sn^{2/3}$, if $n^{-2/3}(t_{j+1} - t_j)$ converges in distribution to some random variable χ , then Lemma 10 and (59) imply that the ℓ -th largest component completely explored before time $sn^{2/3}$, normalized by $n^{-2/3}$, converges in distribution to $\frac{\chi}{d-1}$. As excursion lengths of $\widehat{Y}_{(n^{2/3}, \cdot)}$ are excursion lengths of $\widehat{Y}_{((d-1)n^{2/3}, \cdot)}$ times $(d-1)$ we learn that the sizes of components discovered before time $sn^{2/3}$ in the exploration process, normalized by $n^{-2/3}$ and ordered, converge in distribution to the ordered excursion sizes of $W^\lambda[0, s]$.

We also need to handle the issue of the simplicity of the resulting graph. The following lemma will be useful and is an immediate consequence of Theorem 1 and 2 of [4].

Lemma 23 *Let $d \geq 3$ and let $\bar{d}_1, \bar{d}_2 \in \{1, \dots, d\}^m$ be degree sequences of length m such that each sequence sum to an even number. Let \mathbf{P}_1 be the distribution of a uniform perfect matching on $\sum_{i=1}^m \bar{d}_1(i)$ vertices, divided to m tuples such that the i -th tuple has $\bar{d}_1(i)$ vertices in it. Similarly define \mathbf{P}_2 using degree sequence \bar{d}_2 . Let $\mathcal{S}imple$ be the event that contracting each tuple into a single vertex yields a simple graph. Assume d is fixed and $m \rightarrow \infty$. If $\bar{d}_1 = (d, \dots, d)$ and \bar{d}_2 has $(1 - o(1))m$ entries with the value d then*

$$\mathbf{P}_2(\mathcal{S}imple) = (1 + o(1))\mathbf{P}_1(\mathcal{S}imple).$$

Fix a real number $s > 0$ and consider a fixed interval $I \subset [0, s]$. Let \mathcal{A}_I denote the event

$$\mathcal{A}_I = \left\{ n^{-2/3} \Phi_m(\widehat{Y}_{(n^{2/3}, \cdot)}) \in I \right\}.$$

For times $t < t'$ denote by $\mathcal{S}[t, t']$ the event that no loops or parallel edges (either closed or open) were found between times t and t' by the exploration process. The closed and open edges inspected by the exploration process are a uniform random matching, hence we have that $\mathbf{P}(\mathcal{S}[0, dn/2]) = \mathbf{P}(\text{Simple})$. After t steps of the exploration process the number of d -tuples with d neutral vertices is at least $n - 2t$. Hence, Lemma 23 shows that if $t = o(n)$ then $\mathbf{P}(\mathcal{S}[t, dn/2] \mid \mathcal{F}_t) = (1 + o(1))\mathbf{P}(\text{Simple})$. Thus, by conditioning on $\mathcal{F}_{sn^{2/3}}$ we find that

$$\mathbf{P}(\mathcal{A}_I \cap \text{Simple}) = (1 + o(1))\mathbf{P}(\mathcal{A}_I)\mathbf{P}(\text{Simple}).$$

Hence, when we condition on Simple , component sizes discovered up to time $sn^{2/3}$, normalized, also converge to excursions of $W^\lambda[0, s]$.

Since we handled only components discovered before time $sn^{2/3}$ for some arbitrary large $s > 0$, our final task for completing the proof is to show that large components are typically found in the beginning of the process, rather the end of it. The next lemma completes the proof of the theorem. \square

Lemma 24 *Let $\mathcal{C}_1^{(sn^{2/3})}$ be the largest component which we started exploring after time $sn^{2/3}$. Then for any $\alpha > 0$ we have*

$$\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(|\mathcal{C}_1^{(sn^{2/3})}| \geq \alpha n^{2/3}) = 0. \quad (60)$$

Proof of Lemma 24. Let $\hat{t}_0 > sn^{2/3}$ be the first time larger than $sn^{2/3}$ at which $A_{\hat{t}_0} = 0$ and let $m = \sum_{k=1}^d \tilde{N}_{\hat{t}_0}^{(k)}$. We continue the exploration process on a graph that has m tuples, of varying sizes between 1 and d , in which the number of k -tuples is $\tilde{N}_{\hat{t}_0}^{(k)}$. After finishing the exploration process we again contract each tuple to a vertex to form the graph G_m^* on the vertex set U of cardinality m . The components discovered before \hat{t}_0 together with G_m^* form $G^*(n, d, p)$. Our analysis will show that from any starting vertex $u \in U$, the drift of the process $\{Y_t\}$ is too small to have components of size $\alpha n^{2/3}$.

Fix some small $\delta > 0$ and denote by \mathcal{A} the event

$$\mathcal{A} = \left\{ \tilde{N}_t^{(d)} \leq n - (1 - 3\delta)t, \quad \tilde{N}_t^{(0)} \geq (1 - 3\delta)pt \quad \forall t \in [sn^{2/3}, \delta n] \right\}.$$

By (19) and (23) we have that for all t

$$\begin{aligned}\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}] &\leq p \frac{\sum_{k=2}^d k(k-1) \tilde{N}_{t-1}^{(d)}}{dn - 2t + 1} - 1 \\ &\leq p \left[\frac{d(d-1) \tilde{N}_{t-1}^{(d)} + (d-1)(d-2)(n - \tilde{N}_{t-1}^{(d)} - \tilde{N}_{t-1}^{(0)})}{dn - 2t + 1} \right] - 1,\end{aligned}$$

where the last inequality is due to the fact that $\sum_{k=2}^{d-1} \tilde{N}_{t-1}^{(k)} \leq (n - \tilde{N}_{t-1}^{(d)} - \tilde{N}_{t-1}^{(0)})$. We now substitute $p = \frac{1+\lambda n^{-1/3}}{d-1}$ and condition on \mathcal{A} . A straightforward calculation gives that for $t \in (sn^{2/3}, \delta n]$, we have

$$\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}, \mathcal{A}] \leq \left(1 + \lambda n^{-1/3}\right) \left(1 - \frac{[(d-2) + O(\delta)]t}{d(d-1)n}\right) - 1.$$

We deduce that if $s = s(\delta, \lambda) > 0$ is large enough, then for all $t \in (sn^{2/3}, \delta n]$,

$$\mathbf{E}[\xi_t \mid \mathcal{F}_{t-1}, \mathcal{A}] \leq -\delta^{-1} n^{-1/3}. \quad (61)$$

Assume we start exploring at time $\hat{t}_0 + 1$ the tuple of a vertex $u \in U$ (i.e., $w_{\hat{t}_0+1}$ is in the tuple corresponding to u). Denote by $\mathcal{C}(u)$ the connected component of u and by γ the stopping time

$$\gamma = \min\{t > 0 : Y_{\hat{t}_0+t} = Y_{\hat{t}_0} - N(w_{\hat{t}_0+1})\}.$$

By bounding $U_j \leq t_j - t_{j-1}$ and $V_j \leq t_j - t_{j-1}$ in Lemma 10 we get

$$|\mathcal{C}(u)| \leq \left(\frac{2}{d-1} + 1\right)\gamma + 1.$$

By optional stopping and (61), since $N(w_{\hat{t}_0+1}) \leq d$, we have that $\mathbf{E}[\gamma \wedge \delta n \mid \mathcal{A}] \leq \delta d n^{1/3}$ as long as s is large enough. By (32) and (33) of Lemma 15 we have that for n large enough $\mathbf{P}(\mathcal{A}^c) \leq n^{-1}$. Also, part 1 of Theorem 2 implies that $\mathbf{P}(\gamma > \delta n) \leq n^{-1}$ for large enough n . Hence,

$$\begin{aligned}\mathbf{E} \gamma &\leq dn \mathbf{P}(\gamma > \delta n) + \mathbf{E}[\gamma \mathbf{1}_{\{\gamma \leq \delta n\}}] \leq d + \mathbf{E}[\gamma \wedge \delta n] \\ &\leq d + \mathbf{E}[\gamma \wedge \delta n \mid \mathcal{A}] + \delta n \mathbf{P}(\mathcal{A}^c) \leq (d+1)\delta n^{1/3},\end{aligned}$$

for large enough $s > 0$. The same analysis works for any $u \in U$ and so we learn that $\mathbf{E}|\mathcal{C}(u)| \leq O(\delta)n^{1/3}$ for all $u \in U$. Thus for any fixed $\alpha > 0$ we have

$$\mathbf{P}(|\mathcal{C}(u)| > \alpha n^{2/3}) \leq O(\delta)n^{-1/3},$$

where the constants in the O-notation depend on α and d .

Let X be the random variable counting the number of $u \in U$ such that $|\mathcal{C}(u)| > \alpha n^{2/3}$. As $m \leq n$ we have proved that $\mathbf{E} X \leq O(\delta) n^{2/3}$. Observe that $|\mathcal{C}_1^{(sn^{2/3})}| > \alpha n^{2/3}$ implies that $X > \alpha n^{2/3}$. Hence

$$\mathbf{P}\left(|\mathcal{C}_1^{(sn^{2/3})}| > \alpha n^{2/3}\right) \leq O(\delta).$$

Since $\delta > 0$ was arbitrary and s was large enough depending only on δ and λ , this concludes our proof. \square

We now turn to the proof of Theorem 21. For the proof we use a standard functional central limit theorem for martingales (see [12], Theorem 7.2):

Theorem 25 *Let*

$$\{X_{m,k}, \mathcal{F}_{m,k} : 1 \leq k \leq m\},$$

be a martingale difference array. For any $\ell \leq m$ let

$$V_{m,\ell} = \sum_{k=1}^{\ell} \mathbf{E}[X_{m,k}^2 \mid \mathcal{F}_{m,k-1}]$$

be the quadratic variation process, and

$$Z_{m,\ell} = \sum_{k=1}^{\ell} X_{m,k}.$$

If

1. $|X_{m,k}| \leq \delta_m$ with $\delta_m \rightarrow 0$, and
2. for each $t \in [0, 1]$ we have $V_{m, \lfloor mt \rfloor} \rightarrow t$ in probability as $m \rightarrow \infty$,

then $Z_{m,(\lfloor mt \rfloor)} \xrightarrow{d} B(t)$, where $B(\cdot)$ is standard Brownian motion, and $Z_{m,(\cdot)}$ is the continuous linear interpolation of $Z_{m,k}$.

Proof of Theorem 21. Since $|n^{-1/3}Y_u - n^{-1/3}\widehat{Y}_u| \leq dn^{-1/3}$ for all $u \geq 0$, it suffices to prove the convergence for the process $\{Y_u\}$. Fix some $s > 0$, take $m = m_n = \lfloor sn^{2/3} \rfloor$ and denote $\xi_k^* = \mathbf{E}[\xi_k \mid \mathcal{F}_{k-1}]$. Consider the martingale difference array,

$$X_{m,k} = m^{-1/2}(\xi_k - \xi_k^*), \quad k \leq m.$$

We have that for any $\ell \leq m$,

$$Z_{m,\ell} = \sum_{k=1}^{\ell} X_{m,k} = m^{-1/2}Y_{\ell} - m^{-1/2} \sum_{k=1}^{\ell} \xi_k^*. \quad (62)$$

As $|X_{m,k}| = O(n^{-1/3})$, condition 1 of Theorem 25 is satisfied. Putting $\varepsilon = \lambda n^{-1/3}$ in (i) and (ii) of Corollary 14 gives that

$$\sup_{k \leq m_n} |\xi_k^*| \rightarrow 0 \quad \text{in } L_1 \text{ and in probability.}$$

Hence by (iii) of Corollary 14 we get

$$\sup_{k \leq m_n} \mathbf{E} [(\xi_k - \xi_k^*)^2 \mid \mathcal{F}_{k-1}] \rightarrow (d-2) \quad \text{in probability,}$$

as $n \rightarrow \infty$. Thus for any $t \in [0, 1]$ we have

$$m^{-1} \sum_{k=1}^{\lfloor mt \rfloor} \mathbf{E} [(\xi_k - \xi_k^*)^2 \mid \mathcal{F}_{k-1}] \rightarrow (d-2)t \quad \text{in probability.}$$

In the notation of Theorem 25, it follows that $V_{m, \lfloor mt \rfloor} \rightarrow (d-2)t$ in probability. We conclude by Theorem 25 that

$$Z_{m, (mt)} \xrightarrow{d} B((d-2)t).$$

An immediate computation with Part (ii) of Corollary 14 shows that

$$\mathbf{E} \sum_{k=1}^m |\xi_k^* - \mathbf{E} \xi_k| = O(1),$$

which by the triangle inequality gives that

$$m^{-1/2} \mathbf{E} \max_{k_0 \leq m} \left| \sum_{k=1}^{k_0} \xi_k^* - \sum_{k=1}^{k_0} \mathbf{E} \xi_k \right| = O(n^{-1/3}). \quad (63)$$

Part (i) of Corollary 14 with $\varepsilon = \lambda n^{-1/3}$ implies that for any $t \in [0, 1]$,

$$m^{-1/2} \sum_{i=0}^{tm} \mathbf{E} \xi_i \longrightarrow \lambda t \sqrt{s} - \frac{(d-2)t^2 s^{3/2}}{2d(d-1)}.$$

We conclude by (63) that

$$\mathbf{P} \left(\sup_{t \in [0, 1]} \left| m^{-1/2} \sum_{k=1}^{\lfloor tm \rfloor} \xi_k^* - \lambda \sqrt{s} t + \frac{(d-2)s^{3/2} t^2}{2d(d-1)} \right| > n^{-1/6} \right) \longrightarrow 0. \quad (64)$$

Rearranging (62) using (64) gives that for any fixed $s > 0$

$$\frac{n^{-1/3}}{\sqrt{s}} Y_{(n^{2/3}ts)} \xrightarrow{d} B((d-2)t) + \lambda t \sqrt{s} - \frac{(d-2)t^2 s^{3/2}}{2d(d-1)}.$$

Multiplying by \sqrt{s} and using Brownian scaling gives

$$n^{-1/3} Y_{(n^{2/3}ts)} \xrightarrow{d} B((d-2)ts) + \lambda ts - \frac{(d-2)(ts)^2}{2d(d-1)}.$$

By Brownian scaling and the definition of B^λ we deduce that

$$n^{-1/3} Y_{((d-1)n^{2/3}\cdot)} \xrightarrow{d} (d-1)B^\lambda(\cdot),$$

which concludes our proof. \square

9 Concluding Remarks

- It is natural to ask whether the bounds in Proposition 1 are tight. In light of Theorem 2 we would expect that for $\lambda \in \mathbb{R}$ there exists a constant $c = c(\lambda)$ such that for *any* d -regular graph G and $A > 0$ we have

$$\mathbf{P}\left(|\mathcal{C}_1(G_p)| > An^{2/3}\right) \leq e^{-cA^3}.$$

The authors currently know how to prove this for some particular cases, for instance, expander graphs.

- It is an interesting topic for further research to find a quenched version of Theorem 5. Recall that $|\gamma_1|$ is the longest excursion above past minima of the process B^λ defined in (6). Let $D(n, d)$ denote the number of simple d -regular graphs on n vertices and set $p = \frac{1+\lambda n^{-1/3}}{d-1}$. We expect that for small $\varepsilon_1 > 0, \varepsilon_2 > 0$ and any $s > 0$ and n large enough at least $(1 - \varepsilon_1)D(n, d)$ of the d -regular graphs G on n vertices satisfy

$$\left| \mathbf{P}(|\mathcal{C}_1(G_p)| < sn^{2/3}) - \mathbf{P}(|\gamma_1| \leq s) \right| \leq \varepsilon_2.$$

- Assume now $d = d(n)$ grows with n . We proved that when $d(n)$ is a fixed constant, then $G(n, d(n), p)$ is mean field around $\frac{1}{d(n)-1}$. The same result holds for $d(n) = n - 1$ since this is just the usual $G(n, p)$ model. It seems plausible that for all such sequences (assuming $nd(n)$ is even) the same conclusion still holds.

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